Processus temps discrets, Mouvement Brownien et Evaluation des Actifs Contingents
Partie Evaluation des Actifs Contingents

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1. Introduction

- Financial Asset: Firstly, a contract, which only generate flows of money is a financial assets. Secondly, a contract, which only generate flows of other financial assets is also a financial asset.
- Financial Derivative: Financial asset, whose price only depends on the value of other more basic underlying variables, such as Stock prices, Bond prices, Temperature, Snow depth, ...
- (Non-financial) Derivatives $\exists$ since thousands of years: Forward Contracts on raw products, s.a. wheat
- Synonyms in this course: Financial Derivative, Derivative, Derivative Security, Contingent Claim, ...
- Fundamental Problem: Find a fair price of a derivative (evaluation). Black, Merton and Scholes ($\sim$1973)

- Role of Derivatives:
  - Hedge (cover) risks for some
  - Portfolio management and speculation for others
  - Arbitraging for a small number

1.1 Some different types of derivatives

Underlying Assets: Spot price of a stock, Future price of a stock, Exchange rate ($/€, ...), ... Underlying Assets also called Primary Assets

A) Forward

Forward (contract): Contract that stipulates that its holder can and shall buy the underlying for a predetermined amount K (the delivery price) at a given future time T (the time of maturity). The delivery price K is determined such that the value of the contract is zero at the contract date

Forward Price: Let t_0 be the contract date. Then, by definition, the Forward Price of the underlying at t_0 for delivery at T is K.

B) Options

- European Call Option: Contract that stipulates that its holder can buy the underlying for a predetermined amount K (the strike) at a given future time T (the time of maturity or exercise date).
- American Call Option: As European except that all exercises dates $t \leq T$ are allowed.
- Put Options: Substitute sell in place of buy in def. of Call
- Pay-Off at exercise time t_e and price of underlying S_{t_e} :
  - Call: $(S_{t_e} - K)_+$
  - Put: $(K - S_{t_e})_+$
- Other Options: Asian, Barrier, Caps, Floors, Swaptions, Straddle, Bermudan, Russian, ...

1.2 Market Models

- Mono-Period Models: Trading dates $\mathbb{T} = \{0, T\}$,

- Discrete Time Models: Trading dates $\mathbb{T} = \{0, 1, \ldots, T-1, T\}$,

- Continuous Time Models: Trading dates $\mathbb{T} = [0, T]$ (Most simple and realistic),

1.3 Arbitrage Pricing

- An Arbitrage Portfolio in a financial market (with a risk-free asset) is a self-financing portfolio θ, whose value $V_t(\theta)$ at date $t \in \mathbb{T}$ satisfies:
  i) $V_0(\theta) = 0$
  ii) $V_T(\theta) \geq 0$
  iii) $P(V_T(\theta) > 0) > 0$
- Arbitrage Free Market: $\nexists$ an arbitrage portfolio; AOA
- Arbitrage Pricing of Derivatives in a arbitrage free market of underlying assets: The price of a derivative is determined such that the extended market of underlying assets and the derivative is Arbitrage Free

Efficient Market Hypothesis: Asset prices reflect all information and no one can earn excess returns with certainty.
AOA is a precise mathematical formulation of the Efficient Market Hypothesis.

1.4 Frictionless and Ideal Market

We make several simplifying hypotheses concerning the financial market (frictionless market):

- The number of financial assets is constant in time
- Asset prices take real values. No dividends are payed
- One can sell and buy any real number multiple of an asset
- Buy and sell prices are equal (i.e. no transaction costs)
- Lend and borrow interest rates are equal
- The price of asset is independent of the amount one buy and sell (Price-taker market)
- All information is public

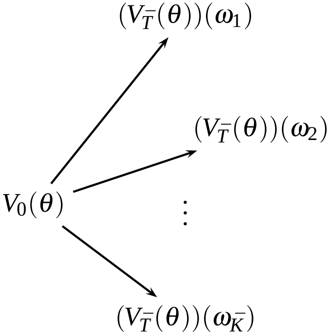
A more realistic market can be obtained by modifying the Ideal Market, with transaction costs and other frictions

2. Mono-Period Market
2.1 Probabilistic Model

- Trading dates: $\mathbb{T} = \{0, T\}$
- Probability space: $(\Omega, P, \mathcal{F})$, Ω set of elementary events, P a priori probability, $\mathcal{F}$ σ-algebra of possible events
- Usually, but not always: $\Omega = \{\omega_1, \dots, \omega_{\bar{K}}\}$, where $\bar{K}$ is the number of elementary events, $p_i = P(\omega_i) > 0$, $\mathcal{F} = \{\text{subsets of } \Omega\}$
- N general Assets with prices $S^1, \dots, S^N$ (Quoted Spot Prices) and possibly one more asset, a risk-free Asset with deterministic price S^0.
- Spot prices at $t = 0$: $S_0 = (S_0^1, \dots, S_0^N) \in \mathbb{R}^N$ if N assets; $S_0 = (S_0^0, S_0^1, \dots, S_0^N) \in \mathbb{R}^{N+1}$ and $S_0^0 > 0$, if $N + 1$ assets. By default $S_0^0 = 1$, if not specified differently.

- Spot prices at T : $S_T = (S^1_T, \ldots, S^N_T)$ is a random vector in $\mathbb{R}^N$ if N assets; $S_T = (S^0_T, S^1_T, \ldots, S^N_T)$ is a random vector in $\mathbb{R}^{N+1}$ if $1+N$ assets, $S^0_1 > 0$,
- Interest Rate r, when S^0 $\exists$: $S^0_T = (1+r)S^0_0$, so $1+r>0$
- Portfolio: θ^i is the number of units of the i-th asset held in the portfolio θ. $\theta = (\theta^1, \ldots, \theta^N) \in \mathbb{R}^N$ if N assets. $\theta = (\theta^0, \ldots, \theta^N) \in \mathbb{R}^{1+N}$ if $1+N$ assets
- Value $V(\theta)$ of a prtf θ:
  - at $t=0$: $V_0(\theta) = \sum_i \theta^i S^i_0 = \theta \cdot S_0 \in \mathbb{R}$
  - at $t=T$: $V_T(\theta) = \sum_i \theta^i S^i_T = \theta \cdot S_T$ is random variable in $\mathbb{R}$
- Gains from date 0 to date T on the investment $V_0(\theta)$ in the prtf θ: $G(\theta) = V_T(\theta) - V_0(\theta) = \theta \cdot (S_T - S_0)$ is random variable in $\mathbb{R}$
- Return on the investment in the prtf θ: $R(\theta) = V_T(\theta)/V_0(\theta)$ if $V_0(\theta) \neq 0$

- Discounted prices, gains and return, when risk-free asset with price S^0 $\exists$: $\bar{S}^i_t = S^i_t / S^0_t$, $\bar{V}_t(\theta) = \theta \cdot \bar{S}^0_t$, $\bar{G}(\theta) = \bar{V}_T(\theta) - \bar{V}_0(\theta)$, $\bar{R} = \bar{V}_T(\theta) / \bar{V}_0(\theta)$
- Representation when Ω is a finite set:



- Matrix notation for calculations when Ω is a finite set:
  - Let $\mathbf{S}$ and $\mathbf{R}$ be, when risk-free asset with price $S^0 \exists$ the $\bar{K} \times (1+N)$ matrix and when risk-free asset with price $S^0 \nexists$ the $\bar{K} \times N$ matrix, with elements $\mathbf{S}_{ij} = S^j_T(\omega_i)$ and $\mathbf{R}_{ij} = \mathbf{S}_{ij}/S^j_0$ respectively. $\mathbf{S}$ is the price matrix and $\mathbf{R}$ the matrix of returns.
  - Given a prtf θ, with $V_0(\theta)) \neq 0$. Let Θ and ϑ be, when risk-free asset with price $S^0 \exists$ the $(1+N) \times 1$ matrix and when risk-free asset with price $S^0 \nexists$ the $N \times 1$ matrix, with elements $\Theta_i = \theta^i$ and $\vartheta_i = \theta^i S^i_0 / V_0(\theta))$ respectively. Θ is the portfolio matrix and ϑ the portfolio fractions matrix.
  - Let $\mathbf{V}$ be the $\bar{K} \times 1$ matrix with elements $\mathbf{V}_i = (V_T(\theta))(\omega_i)$.
  - Then $\mathbf{V} = \mathbf{S}\Theta$ and $(R(\theta))(\omega_i) = (\mathbf{R}\vartheta)_i$ for $1 \leq i \leq \bar{K}$.

2.2 Arbitrage and Equivalent Martingale Measure

- An Arbitrage Portfolio (or an Arbitrage Opportunity) is a portfolio θ such that one of the following two statements A and B is true:

  A: The following three statements are true
  - i) $V_0(\theta) = 0$
  - ii) $V_T(\theta) \geq 0$
  - iii) $P(V_T(\theta) > 0) > 0$

  B: The following two statements are true
  - i) $V_0(\theta) < 0$
  - ii) $V_T(\theta) \geq 0$

- Caution: In this course the above definition is specific for mono-period case
- An Arbitrage Free Market is a market where $\nexists$ an Arbitrage Portfolio; (Also: Arbitraged Market, AOA)

• State Price Vector: A vector $\beta = (\beta_1, \ldots, \beta_{\bar{K}}) \in \mathbb{R}^{\bar{K}}$ satisfying $\forall i$

$$S_0^i = \sum_{1 \leq j \leq \bar{K}} S_T^i(\omega_j)\beta_j \tag{1}$$

is called a State Price Vector. The number β_i is called a state price (of ω_i)

• Arrow-Debreu assets with pay-off $e_1, \ldots, e_{\bar{K}}$: e_i has pay-off 1 in the state ω_i and pay-off 0 in ω_j if $i \neq j$, i.e. $e_i(\omega_j) = \delta_{ij}$, $\forall i, j \in \{1, \ldots, \bar{K}\}$.

• Interpretation of β_i: If, for a given i, e_i is one of the primary assets, then (1) gives that β_i is the price of e_i at $t = 0$

• Price $\Pi_0(X)$ at date $t = 0$ of a general pay-off X at date T.
Intuitively, since β_i not always tradable: An asset with pay-off $X(\omega_i)e_i$ at date T has price $X(\omega_i)\beta_i$ at $t = 0$, so a price candidate is (justification in §3)

$$\Pi_0(X) = \sum_{1 \leq i \leq \bar{K}} X(\omega_i)\beta_i \qquad (2)$$

Caution: β not unique in general $\Rightarrow \Pi_0(X)$ not unique in general

Theorem 2.1

$\nexists$ *an arbitrage portfolio if and only if* $\exists\ \beta$ *such that* $\beta_i > 0$, $\forall$ $1 \leq i \leq \bar{K}$. **Proof:** □

- Equivalent martingale Measure (E.M.M.) and Interest Rate:
  - Suppose that $\exists$ state price vector β, with $\beta_i > 0\ \forall i$. Define r by
$$1/(1+r) = \sum_{1 \le i \le \bar{K}} \beta_i \tag{3}$$
  - r is the interest rate defined by a risk-free asset. In fact, Eq. (2) $\Rightarrow$
$$\Pi_0(1+r) = \sum_{1 \le i \le \bar{K}} (1+r)\beta_i = 1 \tag{4}$$
  - Define $q_i = (1+r)\beta_i$ and the measure Q by $Q(\omega_i) = q_i$. Then $q_i > 0$ and $\sum_{1 \le i \le \bar{K}} q_i = 1$, so Q is a probability measure and $P \sim Q$.

- Expected value w.r.t. Q is: $E_Q[X] = \sum_{1 \leq i \leq \bar{K}} X(\omega_i) q_i$. Eq. (2) $\Rightarrow$

$$\Pi_0(X) = E_Q\left[\frac{X}{1+r}\right] \tag{5}$$

- Ingredients : Arbitrage Price, $Q \sim P$ and Pay-Off discounted to $t = 0$

- Interpretation of Q, when risk-free asset with price S^0 $\exists$:
  - $\bar{S}$, the discounted prices are defined by: $\bar{S}^i_t = S^i_t / S^0_t$ for $t \in \mathbb{T}$. So $\bar{S}^0_0 = \bar{S}^0_T = 1$
  - Eq. (5) gives

$$\bar{S}^i_0 = E_Q\left[\bar{S}^i_T\right], \quad 0 \leq i \leq N \tag{6}$$

  - Eq. (6) $\Rightarrow$ $\bar{S}$ is a martingale under Q
- Definition of e.m.m. (in market with or without S^0):

Definition 2.2

An Equivalent Martingale Measure (e.m.m.) Q is a probability measure on $(\Omega, \mathcal{F})$ equivalent to P, such that for some $r > -1$ and $\forall i$

$$S^i_0 = E_Q\left[\frac{S^i_T}{1+r}\right]. \tag{7}$$

i) In general there is not a unique e.m.m., i.e. equation (7) for Q and r has not always a unique solution.
ii) If Q and r is a solution of (7), then $\beta_i = q_i/(1+r)$, with $q_i = Q(\omega_i)$, defines a state price vector β satisfying (3).

- First Fundamental Theorem of Finance: Theorem 2.1 gives

Theorem 2.3

$\exists$ an Equivalent Martingale Measure if and only if the market is arbitrage free

Corollary 2.4

In an arbitrage free market, we have for all interest rates r and e.m.m. Q solution of (7) and all portfolios θ that

$$V_0(\theta) = E_Q\left[\frac{V_T(\theta)}{1+r}\right]. \tag{8}$$

- $\bar{V}_t(\theta)$ portfolio price discounted to date 0 if S^0 $\exists$: Let $\bar{V}_t(\theta) = V_t(\theta)/S_t^0$. Corollary 2.4 $\Rightarrow$

$$\bar{V}_0(\theta) = E_Q\left[\bar{V}_T(\theta)\right]. \tag{9}$$

2.3 Pricing of Derivative Products

We here introduce arbitrage pricing methods clarifying the validity and meaning of pricing formulas (2) and (5).

• Derivative Product: A derivative is defined by its pay-off X at date T, where X is any $\mathcal{F}$-measurable r.v. (Only mono-period case)

• Hedging Portfolio: A derivative X is *hedgeable* (or *attainable*) if there $\exists$ a prtf. θ s.t.

$$V_T(\theta) = X. \tag{10}$$

Such θ is called a *Hedging portfolio* of X.

• $\mathcal{M}$: In the sequel $\mathcal{M}$ denotes the spot market defined by S and the set of possible portfolios.

Example 2.5

We consider the mono-period market with interest rate $r = 10\%$ and two stocks S^1 et S^2 :

- Price at $t = 0$: $S_0^1 = S_0^2 = 100$
- Price at $t = T$:

$$S_T^1(\omega_1) = 88, \quad S_T^1(\omega_2) = 110, \quad S_T^1(\omega_3) = 132, \tag{11}$$

$$S_T^2(\omega_1) = 132, \quad S_T^2(\omega_2) = 88, \quad S_T^2(\omega_3) = 110. \tag{12}$$

Find a hedging portfolio of a Put on S^1 with strike 105.

Solution:
The pay-off is $X = (105 - S^1_T)_+$, so $X(\omega_1) = 17$, $X(\omega_2) = 0$ and $X(\omega_3) = 0$. The hedging portfolio θ shall satisfy $V_T(\theta) = X$.
In matrix notation

$$\mathbf{X} = \mathbf{S}\Theta, \tag{13}$$

where

$$\mathbf{X} = \begin{pmatrix} 17 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{S} = \begin{pmatrix} \frac{11}{10} & 88 & 132 \\ \frac{11}{10} & 110 & 88 \\ \frac{11}{10} & 132 & 110 \end{pmatrix}. \tag{14}$$

S is invertible, which gives

$$\Theta = \mathbf{S}^{-1}\mathbf{X} = \begin{pmatrix} \frac{170}{33} \\ -\frac{17}{66} \\ \frac{17}{66} \end{pmatrix}. \tag{15}$$

So the hedging portfolio is $\theta = (\frac{170}{33}, -\frac{17}{66}, \frac{17}{66})$ and its price at $t = 0$ is $V_0(\theta) = \frac{170}{33}$.
Fin Example 2.5.

• The value of Hedging Portfolios at $t = 0$ of X is unique:
Suppose that the market $\mathcal{M}$ is Arbitrage Free. Let X be a hedgeable derivative and let θ and η be hedging portfolios of X. Then $V_0(\theta) = V_0(\eta)$. Let Q and r satisfy Eq. (7). Then

$$V_0(\theta) = E_Q\left[\frac{X}{1+r}\right]. \tag{16}$$

Proof: Follows from (8) of Corollary 2.4. □

• $\mathcal{M}'$: Let X be a derivative and $x \in \mathbb{R}$. In next theorem $\mathcal{M}'$ denotes the market with prices S_0 and x at $t = 0$ and prices S_T and X at $t = T$.

Theorem 2.6

Let $\mathcal{M}$ be Arbitrage Free. Then the following three statements are equivalent:

i) *The market $\mathcal{M}'$ is arbitrage free*

ii) $\exists$ *an e.m.m. Q and a interest rate r, satisfying Eq. (7) and*

$$x = E_Q\left[\frac{X}{1+r}\right]. \tag{17}$$

iii) $\exists$ β, *satisfying (1), s.t.* $\beta_i > 0$, $\forall\, 1 \le i \le \bar{K}$ *and s.t.*

$$x = \sum_{1 \le i \le \bar{K}} X(\omega_i)\beta_i. \tag{18}$$

Corollary 2.7

Let $\mathcal{M}$ be Arbitrage Free. If X is hedgeable, then the price x for which statement ii) of Theorem 2.6 holds true is unique.

• Thus, for a hedgeable derivative X, Theorem 2.6 and Corollary 2.7 justify to call this unique price x, *The Arbitrage Price* of X. It is denoted $\Pi_0(X)$ and

$$\Pi_0(X) = E_Q\left[\frac{X}{1+r}\right], \tag{19}$$

for any e.m.m. Q and a interest rate r, satisfying Eq. (7).

Example 2.8

We consider the mono-period market with interest rate $r = 5\%$, two stocks S^1 and S^2 and three states ω_1, ω_2 and ω_3 :

$$\frac{S_T^1}{S_0^1}(\omega_1) = \frac{42}{31}, \ \frac{S_T^1}{S_0^1}(\omega_2) = \frac{21}{31}, \ \frac{S_T^1}{S_0^1}(\omega_3) = \frac{21}{62}$$

and

$$\frac{S_T^2}{S_0^2}(\omega_1) = \frac{21}{124}, \ \frac{S_T^2}{S_0^2}(\omega_2) = \frac{42}{31}, \ \frac{S_T^2}{S_0^2}(\omega_3) = \frac{168}{31}.$$

The a priori probability of ω_1, ω_2 and ω_3 are $1/8$, $3/8$ and $4/8$ respectively.
What is the price (at $t = 0$) of a Call on S^2 with strike $(150/31)S_0^2$, obtained by using an e.m.m. Q? Also, find a hedging portfolio of the Call. What is the price of the hedging portfolio?

Solution:

Let $q_k = Q(\{\omega_k\})$. The eq. $E_Q\left[\bar{S}_T\right] = \bar{S}_0$ gives $E_Q\left[S_T\right] = (1+r)S_0$, which then gives

$$q_1 S_T^i(\omega_1) + q_2 S_T^i(\omega_2) + q_3 S_T^i(\omega_3) = (1+r)S_0^i \quad i = 0, 1, 2.$$

With matrix notation we obtain

$$(\mathbf{R})^t \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = (1+r) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{20}$$

Here

$$\mathbf{R} = \begin{pmatrix} \frac{21}{20} & \frac{42}{31} & \frac{21}{124} \\ \frac{21}{20} & \frac{21}{31} & \frac{42}{31} \\ \frac{21}{20} & \frac{21}{62} & \frac{168}{31} \end{pmatrix}. \tag{21}$$

The unique solution of (20) is

$$q_1 = \frac{3}{5},\ q_2 = \frac{3}{10},\ q_3 = \frac{1}{10}. \tag{22}$$

The pay-off of the Call is $X = (S_T^2 - \frac{150}{31} S_0^2)_+$, so $X(\omega_1) = 0$, $X(\omega_2) = 0$ and $X(\omega_3) = \frac{18}{31} S_0^2$. Its price at $t = 0$ is

$$E_Q\left[\frac{X}{1+r}\right] = q_3 \frac{X(\omega_3)}{1+r} = \frac{12}{217} S_0^2. \tag{23}$$

A hedging portfolio θ of the Call shall satisfy $V_T(\theta) = X$. In matrix notation

$$\mathbf{X} = \mathbf{S}\Theta = \mathbf{R}\begin{pmatrix}\theta^0 S_0^0 \\ \theta^1 S_0^1 \\ \theta^2 S_0^2\end{pmatrix}, \quad \text{where} \quad \mathbf{X} = \begin{pmatrix}0 \\ 0 \\ \frac{18}{31} S_0^2\end{pmatrix}. \qquad (24)$$

This gives

$$\begin{pmatrix}\theta^0 S_0^0 \\ \theta^1 S_0^1 \\ \theta^2 S_0^2\end{pmatrix} = S_0^2 \begin{pmatrix}-\frac{3600}{8897} \\ \frac{12}{41} \\ \frac{48}{287}\end{pmatrix}. \qquad (25)$$

We have $V_0(\theta) = \frac{12}{217} S_0^2$, which, as it should be, is the same as the the price given in (23). **Fin Example 2.8**.

- Complete Market: The market $\mathcal{M}$ is said to be *Complete*, when "all" derivatives are hedgeable.
- Second Fundamental Theorem:

Theorem 2.9

The following two statements are equivalent:

i) *The market $\mathcal{M}$ is arbitrage free and complete*

ii) $\exists$ *a unique e.m.m.* Q.

Corollary 2.10

In an arbitrage free and complete market, every derivative X has a unique arbitrage price $\Pi_0(X)$ at $t = 0$, given by (19).

• Pricing of a derivative X in an Incomplete Market: If X is hedgeable, then a unique arbitrage price $\Pi_0(X)$ is given by (19) according to Corollary 2.7. As we shall see, there is no unique arbitrage price, when X is not hedgeable.
• $\mathcal{M}_e$: Let $\mathcal{M}_e$ be the set of e.m.m. for the market $\mathcal{M}$, which is supposed arbitrage free. So $\mathcal{M}_e \neq \emptyset$.
• Possible arbitrage prices of X: Theorem 2.6 and formula (17) gives that for every $Q \in \mathcal{M}_e$ and corresponding interest rate r, a possible arbitrage price is given by

$$E_Q\left[\frac{X}{1+r}\right].$$

This leads to an interval of possible arbitrage prices.

- Lower-Upper price spread: $]\Pi_{*0}(X), \Pi_0^*(X)[$, where

$$\Pi_{*0}(X) = \inf_{Q \in \mathcal{M}_e} E_Q\left[\frac{X}{1+r}\right] \text{ and } \Pi_0^*(X) = \sup_{Q \in \mathcal{M}_e} E_Q\left[\frac{X}{1+r}\right]. \tag{26}$$

Remind that in general r in this formula depends on Q. Let $\mathcal{M}$ be Arbitrage Free and let the price of X at $t = 0$ be x. Then the market $\mathcal{M}'$ is arbitrage free iff $x \in\ ]\Pi_{*0}(X), \Pi_0^*(X)[$.

- Problem: How to choose the price in $]\Pi_{*0}(X), \Pi_0^*(X)[$?

3 Binomial Model, 3.1 Definition

• Assets: One risk-free with price S^0 and one risky asset with price S^1, for ex. a stock

• 2 possible evolutions from t to $t+1$: Up denoted u and Down denoted d. The probabilities of u and d are p and $1-p$ respectively, where $0 < p < 1$ and p is independent of what has happened before t. Typically, if u (resp. d) then the price S^1 evolves by a factor U (resp. D) where $D \leq U$:

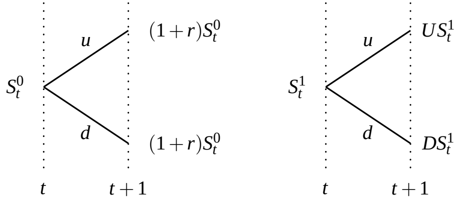


Figure: Price binom. mod.

- The Random Source in Bin. Mod. is a Bernoulli process ν:
  - $\nu = (\nu_1, \dots, \nu_T)$, where the ν_t are i.i.d., $P(\nu_t = 1) = p$ and $P(\nu_t = 0) = 1 - p$
  - $\Omega = \{0,1\}^T$; $\nu_t(\omega)$ is the t:th coordinate of $\omega \in \Omega$, so $\omega = (\nu_1(\omega), \dots, \nu_T(\omega))$. Convention: ω_k has the coordinates given by the binary representation of the integer k, $0 \le k \le 2^T - 1$
  - If $\omega \in \Omega$ corresponds to n Up's (so $T - n$ Down's) then $P(\omega) = p^n(1-p)^{T-n}$
  - Filtration $\{\mathcal{F}_t\}_{t \in \mathbb{T}}$, where $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and $\mathcal{F}_t = \sigma(\nu_1, \dots, \nu_t)$, for $1 \le t \le T$. So $\nu_1, \dots, \nu_t$ are known at t.
  - $\nu_{t+1}(\omega) = 0$ and $\nu_{t+1}(\omega) = 1$ are identified with the evolution d and u respectively, from t to $t+1$.

• Information tree: one-to-one correspondence between states and final leaves

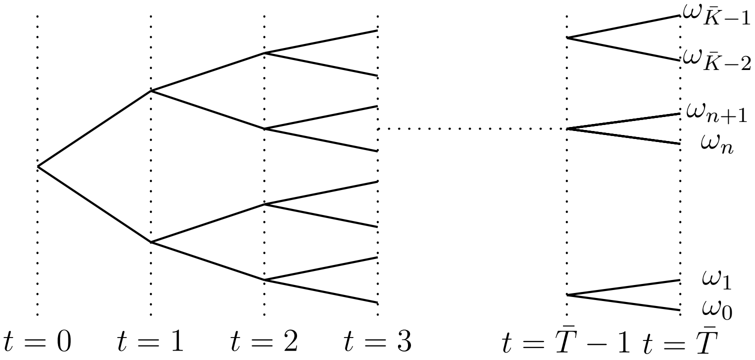


Figure: Information tree; Number of states $\bar{K} = 2^T$

• Path lattice: one-to-one correspondence between states ω and paths

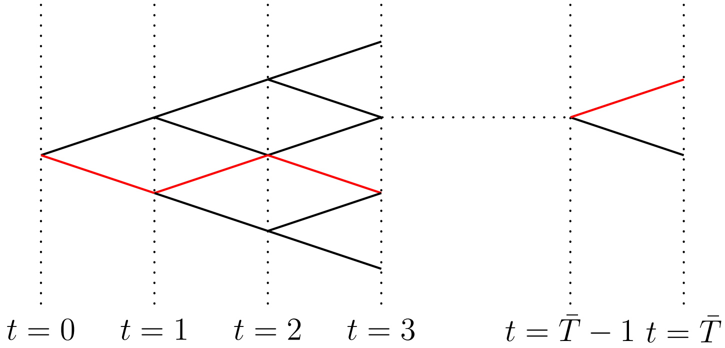


Figure: Path lattice; Number of paths $\bar{K}=2^T$

Example 3.1

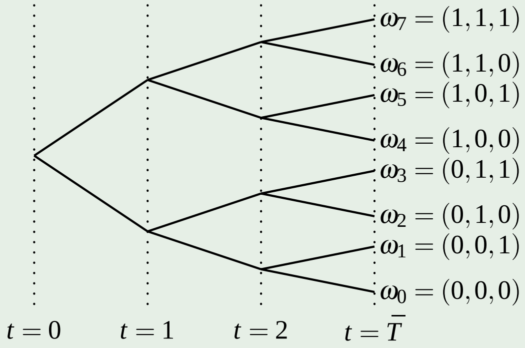


Figure: Information tree, with $T = 3$, $\bar{K} = 8$

2) Filtration:

- $\mathcal{F}_0 = \{\Omega, \emptyset\}$.
- $\mathcal{F}_1 = \sigma(\nu_1)$. To find it, let $A_1(0) = \nu_1^{-1}(\{0\})$ and $A_1(1) = \nu_1^{-1}(\{1\})$, where $\nu_1^{-1}(B)$ is the inverse image of the set B.

$$\text{Then } A_1(0) = \{\omega_0, \omega_1, \omega_2, \omega_3\}, A_1(1) = \{\omega_4, \omega_5, \omega_6, \omega_7\}$$

$$\text{so } \mathcal{F}_1 = \sigma(A_1(0), A_1(1)).$$

Since $\{A_1(0), A_1(1)\}$ is a partition of Ω it follows that, at time $t = 1$ we can distinguish between events in $A_1(0)$ and events in $A_1(1)$, but we can not distinguish between events within $A_1(0)$ or events within $A_1(1)$.

- $\mathcal{F}_2 = \sigma(\nu_1, \nu_2)$. Let $A_2(i,j) = \nu_1^{-1}(\{i\}) \cap \nu_2^{-1}(\{j\})$. Then

$$\begin{aligned} A_2(0,0) = \{\omega_0, \omega_1\}, A_2(0,1) = \{\omega_2, \omega_3\}, \\ A_2(1,0) = \{\omega_4, \omega_5\}, A_2(1,1) = \{\omega_6, \omega_7\}, \end{aligned} \tag{27}$$

  so $\mathcal{F}_2 = \sigma(A_2(0,0), A_2(0,1), A_2(1,0), A_2(1,1)\}$. Since the $A_2(i,j)$ defines a partition of Ω, at $t = 2$ one can distinguish between events which are in two different such sets, but not within the same set.
- $\mathcal{F}_3 = \sigma(\nu_1, \nu_2, \nu_3)$. Let $A_3(i,j,k) = \nu_1^{-1}(\{i\}) \cap \nu_2^{-1}(\{i\}) \cap \nu_3^{-1}(\{k\})$. Then $A_3(i,j,k) = \{(i,j,k)\}$. Explicitly,

$$A_3(0,0,0) = \{\omega_0\}, \dots, A_3(1,1,1) = \{\omega_7\}.$$

  So $\mathcal{F}_3$ is the set of all subsets of Ω.

3) Let X be a r.v. (later on it will be a derivative product)

- X is $\mathcal{F}_0$-measurable, i.e known at $t = 0$, means exactly that X is constant on Ω, i.e. $X(\omega) = X(\omega') \ \forall \omega, \omega' \in \Omega$
- X is $\mathcal{F}_1$-m. $\Leftrightarrow$ X is constant on $A_1(0)$ and constant on $A_1(1)$.
- X is $\mathcal{F}_2$-m. $\Leftrightarrow$ X is constant on each one of the sets $A_2(0,0), A_2(0,1), A_2(1,0), A_2(1,1)$
- X is $\mathcal{F}_3$-m. $\Leftrightarrow$ X is arbitrary

- Number of Up's: Let N_t be the number of Up's up to date t included:

$$N_0 = 0,\ N_t = \nu_1 + \ldots + \nu_t,\ \text{if } t > 0. \tag{28}$$

- Binomial Distribution: If $0 \leq n \leq t$, then

$$P(N_t = n) = \binom{t}{n} p^n (1-p)^{t-n}.$$

- Price Distribution:

$$S_t^1 = S_0^1 U^{N_t} D^{t-N_t}$$

$$\text{gives } P(S_t^1 = S_0^1 U^n D^{t-n}) = \binom{t}{n} p^n (1-p)^{t-n}.$$

- Recall that:

$$E[N_t] = tp \quad \text{and} \quad var[N_t] = tp(1-p).$$

• Markov Process: $\frac{S^1_{t+1}}{S^1_t}$ are independent of $S^1_0, .., S^1_t$

• To sum up, the binomial price model is given by the filtered probability space $(\Omega, P, \mathcal{F}, \mathcal{A})$, the two dimensional price process S and the possible trading times $t \in \mathbb{T} = \{0, \ldots, T\}$, where

- Ω is the set of elementary events
- P is the a priori probability measure
- $\mathcal{F} = \mathcal{F}_T$ is the σ-algebra of all events
- $\mathcal{A} = (\mathcal{F}_t)_{t \in \mathbb{T}}$ is the filtration

Example 3.2

$U = 2$, $D = \frac{1}{2}$, $p = \frac{3}{4}$, $S_0^1 = 1$, $T = 3$ **Reduced tree** S^1 **:**

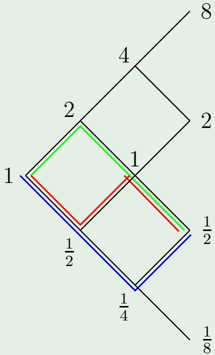
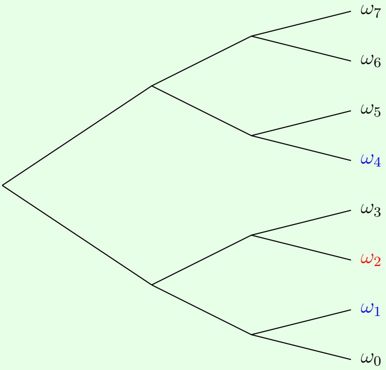


Figure: Price path lattice, $\omega_1 = (0, 0, 1)$, $\omega_2 = (0, 1, 0)$ and $\omega_4 = (1, 0, 0)$.



$$P(\omega_4) = p(1-p)^2 = \frac{3}{4}\left(\frac{1}{4}\right)^2 = \frac{3}{64}$$

$$P(\omega_2) = p(1-p)^2 = \frac{3}{64}$$

$$P(\omega_1) = p(1-p)^2 = \frac{3}{4}\left(\frac{1}{4}\right)^2 = \frac{3}{64}$$

$$P(\omega_0) = (1-p)^3 = \left(\frac{1}{4}\right)^3$$

$$P(S_3^1 = \tfrac{1}{2}) = P(\{\omega_1 \ , \ \omega_2 \ , \ \omega_4\})$$

$$= 3 \cdot \frac{3}{64} = \frac{9}{64}$$

Figure: Information tree

3.2 Portfolios

- A portfolio θ is given by:
  - θ^i_t is the number of units held at time t of asset nr. i
  - $\theta_t = (\theta^0_t, \theta^1_t)$ is the portfolio held at time t. θ_t is known at t, i.e. is $\mathcal{F}_t$-measurable
  - $\theta = (\theta_0, \theta_1, ..., \theta_T)$ is the portfolio process. It is adapted to the filtration $(\mathcal{F}_t)_{t \in \mathbb{T}}$.
- Value (price) $V_t(\theta)$ of θ at t

$$V_t(\theta) = \theta^0_t S^0_t + \theta^1_t S^1_t = \theta_t \cdot S_t$$

- Gains process $G(\theta)$; Gains $G_t(\theta)$ from 0 to t

$$G_t(\theta) = \sum_{0 \leq s < t} \theta_s \cdot (S_{s+1} - S_s)$$

- Discounted prices $\bar{S} = \frac{S}{S^0}$, i.e. $\bar{S}_t^0 = \frac{S_t^0}{S_t^0} = 1$ and $\bar{S}_t^1 = \frac{S_t^1}{S_t^0}$
- Discounted value of θ at t : $\bar{V_T}(\theta) = \frac{V_t(\theta}{S_t^0} = \theta_t \cdot \bar{S}_t$.
- Discounted gains

$$\bar{G}_t(\theta) = \sum_{0 \leq s < t} \theta_s \cdot (\bar{S}_{s+1} - \bar{S}_s)$$

- Self-financed portfolio

Definition 3.3

A portfolio θ is self-financed iff

$$\forall\ t \in \mathbb{T}, \quad V_t(\theta) = V_0(\theta) + G_t(\theta) \tag{29}$$

Theorem 3.4

θ is self-financed iff

$$\forall\ t \in \{0, \dots, T-1\},\ \ \theta_t \cdot S_{t+1} = \theta_{t+1} \cdot S_{t+1} \tag{30}$$

Proof: By definition $V_{t+1}(\theta) = \theta_{t+1} \cdot S_{t+1}$. Suppose (30) true. Repeated use of the following equality then shows that θ is self-financed:

$$V_{t+1}(\theta) = \theta_t \cdot S_{t+1} = \theta_t \cdot S_t + \theta_t \cdot (S_{t+1} - S_t) = V_t(\theta) + \theta_t \cdot (S_{t+1} - S_t)$$

If θ is self-financed then repeated use of this equality, with the last member $= \theta_{t+1} \cdot S_{t+1}$, in the opposite direction proves that (30) is true. ■

Theorem 3.5

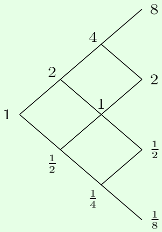
θ is self-financed iff $\forall\ t \in \mathbb{T},\ \ \bar{V}_t(\theta) = V_0(\theta) + \bar{G}_t(\theta)$.

3.3 Arbitrage and Equivalent Martingale Measure (e.m.m.)

Question for motivation of the introduction of e.m.m

- Binomial model $T = 3$, $U = 2$, $D = \frac{1}{2}$, $r = 0$, $p = \frac{3}{4}$, $S_0^1 = 1$
- European Call: Strike $K = 1$ (at the money), Maturity T

$$\text{Pay Off} = X = (S_T^1 - K)_+ = (S_T^1 - 1)_+$$



$7 = X(\{\omega_7\})$

$1 = X(\{\omega_3\}) = X(\{\omega_5\}) = X(\{\omega_6\})$

$0 = X(\{\omega_1\}) = X(\{\omega_2\}) = X(\{\omega_4\})$

$0 = X(\{\omega_0\})$

- What is the price of the Call at $t = 0$?

Answer:

- The price is $\frac{13}{27}$. In fact

$$\frac{13}{27} = Q(S_3^1 = 2) \cdot 1 + Q(S_3^1 = 8) \cdot 7 = 3q^2(1-q) + 7q^3. \quad (31)$$

Q is here an Equivalent Martingale Measure (e.m.m) given by

$$Q(S_{t+1}^1 = US_{t+1}^1) \equiv q = \frac{1+r-D}{U-D} = \frac{1-1/2}{2-1/2} = \frac{1}{3}. \quad (32)$$

- Arbitrage Opportunity (OA) and Arbitrage Portfolio in the Binomial financial market

Definition 3.6

An Arbitrage Portfolio is a self-financing portfolio θ, whose value $V_t(\theta)$ at date $t \in \mathbb{T}$ satisfies:

i) $V_0(\theta) = 0$

ii) $V_T(\theta) \geq 0$

iii) $P(V_T(\theta) > 0) > 0$

- AOA $\Leftrightarrow \nexists$ OA $\Leftrightarrow$ Arbitrage free market

Theorem 3.7

The Binomial financial market is arbitrage free iff

$$D < 1 + r < U \;\; or \;\; D = 1 + r = U.$$

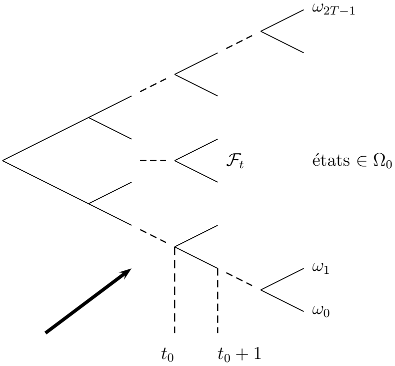
Lemma 3.8

The following two statements are equivalent:

1. *The Binomial market is arbitrage free*
2. *Every mono-period sub-market of the Binomial market is arbitrage free*

Proof:

• Suppose first that there exists a mono-period sub-market with an OA



Sous-Modèle de t_0 à $t_0 + 1$
avec OA permet la construction
d'un portefeuille d'arbitrage de t_0 à $t = T$

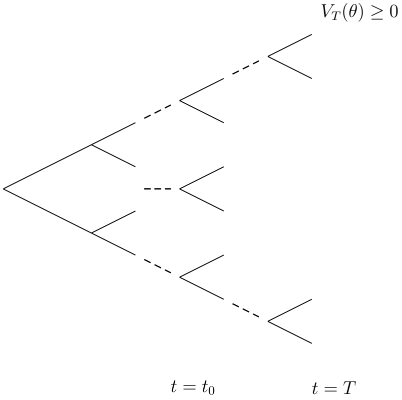
$$\theta_0 = \cdots = \theta_{t_0-1} = 0$$

$$\begin{cases} \theta_{t_0} \cdot S_{t_0} = 0 \\ \theta_{t_0} \cdot S_{t_0+1} \geq 0 \\ E\left[\theta_{t_0} \cdot S_{t_0+1} | \mathcal{F}_t\right] > 0 \end{cases} \tag{33}$$

$\theta_{t_0+1} = (\theta^0_{t+1}, 0) = \cdots = \theta_T = (\theta^0_T, 0)$ all invested in the bank account.

$\Rightarrow \theta$ is an arbitrage portfolio.

• Suppose then that there does not exist a mono-period sub-market with an OA. Let θ be a self-financed portfolio such that $V_T(\theta) \geq 0$.



• Then $V_{T-1}(\theta) \geq 0$, (or otherwise there exists an arbitrage portfolio from $T-1$ to T)
• So, by iteration $V_T(\theta) \geq 0$, $V_{T-1}(\theta) \geq 0$, $\ldots$, $V_0(\theta) \geq 0$.
• Let θ be such that $V_0(\theta) = 0$. Since $V_1(\theta) \geq 0$, we must have $V_1(\theta) = 0$ (if not $\exists$ OA). Similarly

$$V_1(\theta) = 0 \;\; \Rightarrow V_2(\theta) = 0 \;\; \Rightarrow \cdots V_T(\theta) = 0$$

$\Rightarrow \nexists$ arbitrage portfolio θ. ■

• e.m.m.; Equivalent Martingale Measure in the Binomial financial market

Definition 3.9

An e.m.m. is a probability measure Q s.t.
$Q \sim P$ and $\bar{S} = \frac{S}{S^0}$, is a Q martingale, i.e.

$$\bar{S}_t = E_Q \left[\bar{S}_{t+1} | \mathcal{F}_t \right], \quad \text{for all} \quad t = 0, 1, ..., T. \tag{34}$$

- Find an e.m.m; $Q \sim P$ and (34) $\Leftrightarrow$

$$\frac{S_t^1(\omega)}{(1+r)^t} = E_Q\left[\frac{S_{t+1}}{(1+r)^{t+1}}|\mathcal{F}_t\right](\omega)$$

$$\Leftrightarrow S_t^1(\omega) = q_t(\omega)\frac{S_t^1(\omega)U}{(1+r)} + (1-q_t(\omega))\frac{S_t^1(\omega)D}{(1+r)}$$

$$\Leftrightarrow q_t(\omega) = q \equiv \frac{1+r-D}{U-D}, \text{ if } U > D \text{ and } 0 < q_t(\omega) < 1 \text{ if } U = D.$$

Here q is the probability under the mono-periode e.m.m.

Theorem 3.10

For the Binomial financial market there exists an e.m.m. iff

$$D < 1+r < U \text{ or } D = 1+r = U.$$

Corollary 3.11

For the Binomial financial market there exists an e.m.m. iff there is AOA

- Price of a self-financed portfolio θ:

$\bar{V}(\theta)$ is a Q-martingale, i.e.

$$\bar{V}_t(\theta) = E_Q\left[\bar{V}_{t+1}(\theta)|\mathcal{F}_t\right].$$

In fact, since θ is self-financed $\theta_{t+1}\cdot\bar{S}_{t+1} = \theta_t\cdot\bar{S}_{t+1}$ and since θ_t is $\mathcal{F}_t$-measurable:

$$E_Q\left[\bar{V}_{t+1}(\theta)|\mathcal{F}_t\right] = E_Q\left[\theta_{t+1}\cdot\bar{S}_{t+1}|\mathcal{F}_t\right] = E_Q\left[\theta_t\cdot\bar{S}_{t+1}|\mathcal{F}_t\right] \\ = \sum_i \theta_t^i E\left[\bar{S}_{t+1}^i|\mathcal{F}_t\right] = \sum_i \theta_t^i \bar{S}_t^i = \bar{V}_t(\theta). \tag{35}$$

3.4 Price of a derivative and completness

- A European Derivative is a contract, which determines the pay-off $X(\omega)$ for all $\omega \in \Omega$ at the exercise date T. T coincide with the maturity date.

$\Rightarrow$ Bijective relation between real r.v. X on Ω and EU derivatives.

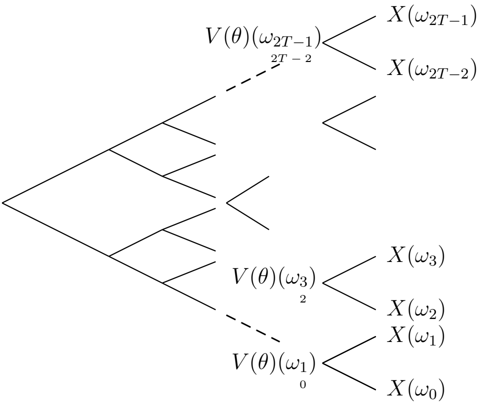
- Hedging portfolio θ of an EU derivative X is a self-financed portfolio θ s.t. $V_T(\theta) = X$. X is said to be hedgeable.
- Consider the cases for which there is AOA

1. $D = 1 + r = U$
2. $D < 1 + r < U$

What are the hedgeable EU derivatives X?

1) $D = 1 + r = U \Rightarrow \; S_t^1 = S_0^1 S_t^0$
$\Rightarrow V_t(\theta) = S_t^0(\theta_t^0 + \theta_t^1 S_0^1)$
θ self-financed $\Rightarrow \; \theta_t S_{t+1} = \theta_{t+1} S_{t+1}$
$\Rightarrow \; \theta_t^0 S_{t+1}^0 + \theta_t^1 S_0^1 S_{t+1}^0 = \theta_{t+1}^0 S_{t+1}^0 + \theta_{t+1}^1 S_0^1 S_{t+1}^0$
$\Rightarrow \; \theta_t^0 + \theta_t^1 S_0^1 = \theta_{t+1}^0 + \theta_{t+1}^1 S_0^1$
$\Rightarrow \; \theta_t^0 + \theta_t^1 S_0^1 = V_0(\theta)$
$\Rightarrow \; V_t(\theta) = V_0(\theta) S_t^0$
$\Rightarrow$ Only deterministic X are hedgeable !!

2) $D < 1 + r < U$. Let X be an arbitrary r.v. on Ω. Try to find θ self-financed and with $V_T(\theta) = X$.



• We have, for $\omega = \omega_0$ or ω_1 :
$V_T(\theta)(\omega) = \theta_T(\omega) \cdot S_T(\omega) = \theta_{T-1}(\omega) \cdot S_T(\omega) = X(\omega)$
But, since θ_{T-1} is $\mathcal{F}_T - 1$-measurable:
$\theta_{T-1}(\omega_0) = \theta_{T-1}(\omega_1) = \theta_{T-1}(\omega) \Rightarrow$

$$\begin{cases} \theta^0_{T-1}(\omega)(1+r)^T + \theta^1_{T-1}(\omega)S^1_{T-1}(\omega)U = X(\omega_1) \\ \theta^0_{T-1}(\omega)(1+r)^T + \theta^1_{T-1}(\omega)S^1_{T-1}(\omega)D = X(\omega_0) \end{cases}$$

$\Rightarrow$

$$\begin{cases} \theta^1_{T-1}(\omega)S^1_{T-1}(\omega) = \frac{X(\omega_1)-X(\omega_0)}{U-D} \\ \theta^0_{T-1}(\omega)(1+r)^{T-1} = \frac{1}{1+r}\frac{X(\omega_0)U-X(\omega_1)D}{U-D} \end{cases}$$

- Same method for $\omega = \omega_2$ or ω_3
- Same method for $\omega = \omega_4$ or ω_5
- etc. same method up to $\omega = \omega_{2T-2}$ or ω_{2T-1}
- Then by iteration from $T-1$ to $T-2$, ..., from $t=1$ to $t=0$.
- $\Rightarrow$ All EU derivatives X are hedgeable.

• Complete Market: A financial market is said to be complete if all derivatives are hedgeable. We have proved

Theorem 3.12

For the Binomial Market the following statements are equivalent

- *The Market is arbitrage free and complete*
- $D < 1 + r < U$
- $\exists$ *a unique e.m.m.* Q

• $\Pi(X)$, Arbitrage price of X: Let θ be a hedging portfolio of X. For such θ, $V(\theta)$ is independent of θ, so we define

$$\forall\ t \in \mathbb{T}\ \ \Pi_t(X) = V_t(\theta). \tag{36}$$

In the sequel price = arbitrage price
• $\bar{\Pi}(X)$, Discounted price of X: $\bar{\Pi}_t(X) = \Pi_t(X)/S_t^0$.

Theorem 3.13

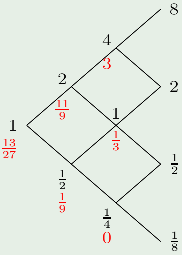
In the Binomial Model the discounted price of any derivative X is a Q-martingale,i.e.

$$\bar{\Pi}_t(X) = E_Q\left[\bar{\Pi}_{t+1}(X) \mid \mathcal{F}_t\right], \quad \textit{for } 0 \le t \le T-1. \tag{37}$$

Example 3.14

Prices for all ω and t in (31)

Price of X



Pay Off $= X = (S_T^1 - K)_+ = (S_T^1 - 1)_+$

$7 = X(\{\omega_7\})$

$1 = X(\{\omega_3\}) = X(\{\omega_5\}) = X(\{\omega_6\})$

$0 = X(\{\omega_1\}) = X(\{\omega_2\}) = X(\{\omega_4\})$

$0 = X(\{\omega_0\})$

Example 3.15 (Barrier)

Let $T = 3$, $r = \frac{1}{10}$, $U = \frac{5}{4}$, $D = \frac{4}{5}$, $S_0^1 = 400$, $p = \frac{3}{4}$. Find the price at $t = 0$ of a Barrier Option of the type Down-And-Out Call with

$$\text{Barrier} \quad H = 350, \quad \text{Strike} \quad K = 450.$$

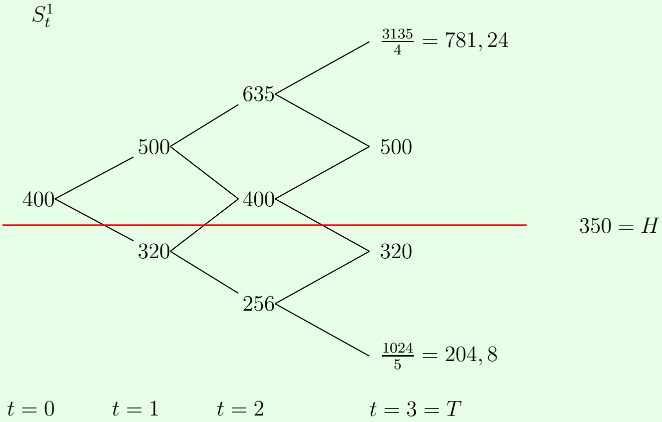
N.B : The Pay Off at T in the state ω is given by

$$X(\omega) = \begin{cases} 0 \text{ if } \min_{0 \le t \le T} S_t^1(\omega) < H \\ (S_T^1(\omega) - K)_+ \text{ if } \min_{0 \le t \le T} S_t^1(\omega) \ge H \end{cases} \tag{38}$$

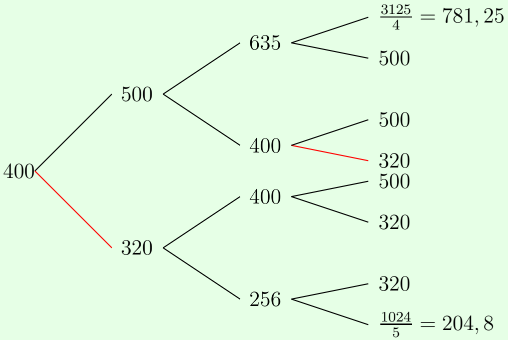
We also have, denoting by $\mathbf{1}_A$ the caracteristic function of a set A :

$$X = (S_T^1 - K)_+ \mathbf{1}_{\{\min_{0 \le t \le T} S_t^1 \ge H\}}. \tag{39}$$

Solution: $q = \frac{1+r-D}{U-D} = \frac{\frac{11}{10}-\frac{4}{5}}{\frac{5}{4}-\frac{4}{5}}$. So $q = \frac{2}{3}$



Information tree with S^1



PAY-OFF in ω

$(\frac{3125}{4} - 450)_+ = \frac{1325}{4}$ (ω_7)

$(500 - 450)_+ = 50$ (ω_6)

50 (ω_5)

0 (ω_4)

0 (ω_3)

0 (ω_2)

0 (ω_1)

0 (ω_0)

- Price at $t = 0$:

$$\Pi_0(X) = E_Q\left[\frac{X}{S_T^0}\right] = \frac{1}{(1+r)^T} E_Q[X]$$

$$\frac{1}{(1+r)^T} E_Q[X] = \left(\frac{10}{11}\right)^3 \left(Q(\{\omega_5\})50 + Q(\{\omega_6\})50 + Q(\{\omega_7\})\frac{1325}{4}\right)$$

$$= \left(\frac{10}{11}\right)^3 \left(2 \cdot 50q^2(1-q) + \frac{1325}{4}q^3\right)$$

$$= \left(\frac{10}{11}\right)^3 \left(100 \cdot \left(\frac{2}{3}\right)^2 (\frac{1}{3}) + \frac{1325}{4} \cdot \left(\frac{2}{3}\right)^3\right)$$

$$= \left(\frac{10}{33}\right)^3 (400 + 2 \cdot 1325) \approx 84.87.$$

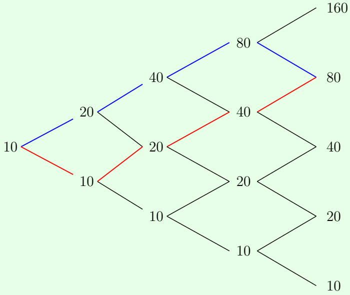
Example 3.16 (Asian Call Option)

Let $T = 4$, $r = \frac{1}{4}$, $D = 1$, $U = 2$, $S_0^1 = 10$, $p = \frac{3}{4}$.
Consider an option with pay-off X at maturity T :

$$X = \left(\frac{1}{T}\sum_{t=1}^{T} S_t^1 - K\right)_+, \quad \text{with strike} \quad K = 35. \tag{40}$$

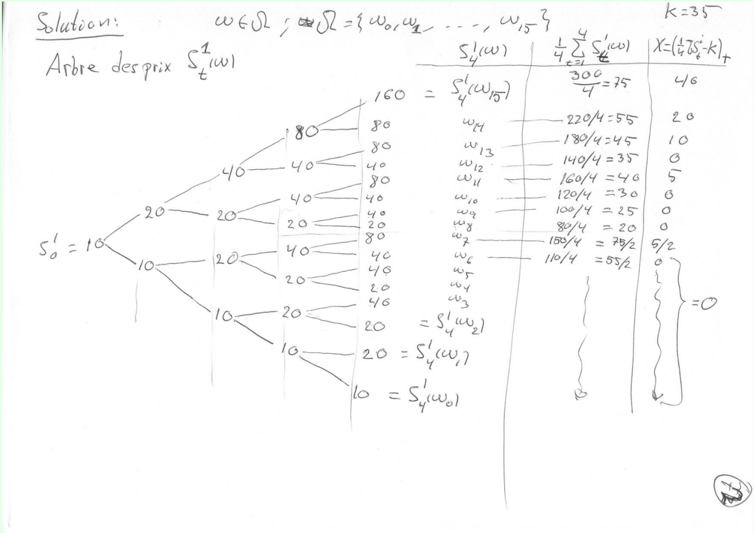
Find the price $(\Pi_X(t))(\omega)$, for all t, ω.

Before studying the price of the option, we note that the pay-off X is path dependant.



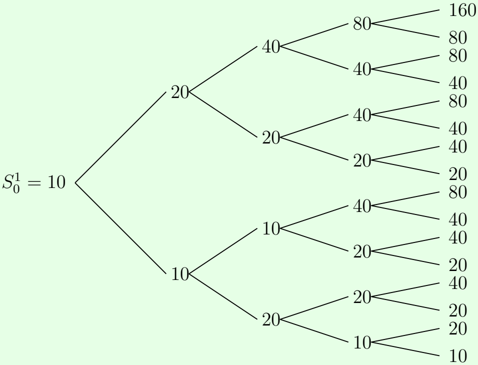
$\Rightarrow \frac{1}{T}\sum_{t=1}^{T} S_t^1 = \frac{220}{4}$ (blue)

$\Rightarrow \frac{1}{T}\sum_{t=1}^{T} S_t^1 = \frac{150}{4}$ (red)


Solution: $\omega \in \Omega$; $\Omega = \{\omega_0, \omega_1, \dots, \omega_{15}\}$
$K = 35$
Arbre des prix $S^1_t(\omega)$
$S^1_4(\omega)$
$\frac{1}{4}\sum_{t=1}^{4} S^1_t(\omega)$
$X = (\frac{1}{4}\sum S^1_t - K)_+$
$S^1_0 = 10$
$160 = S^1_4(\omega_{15})$
$\frac{300}{4} = 75$
46
220/4 = 55
20
180/4 = 45
10
140/4 = 35
0
160/4 = 40
5
120/4 = 30
0
100/4 = 25
0
80/4 = 20
0
150/4 = 75/2
5/2
110/4 = 55/2
0
= 0
$20 = S^1_4(\omega_2)$
$20 = S^1_4(\omega_1)$
$10 = S^1_4(\omega_0)$

$\omega \in \Omega \; ; \;\; \Omega = \{\omega_0, \omega_1, ..., \omega_1 5\} \qquad K = 35$

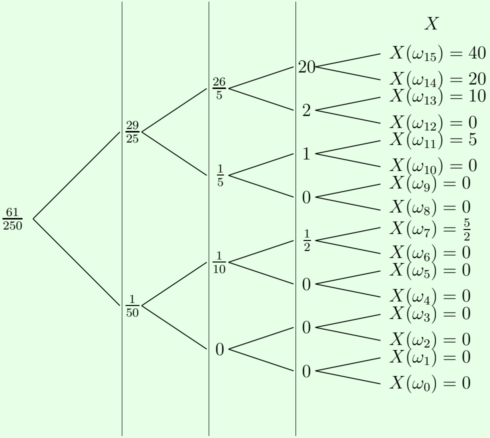
Arbre des prix $S_t^1(\omega)$



$S_4^1(\omega)$	$\frac{1}{4}\sum_{t=1}^4 S_t^1(\omega)$	$X = (\frac{1}{4}\sum_{t=1}^4 S_t^i - K)_+$
$S_4^1(\omega_{15}) = 160$	$\frac{300}{4} = 75$	$X(\omega_{15}) = 40$
$S_4^1(\omega_{14}) = 80$	$\frac{220}{4} = 55$	$X(\omega_{14}) = 20$
$S_4^1(\omega_{13}) = 80$	$\frac{180}{4} = 45$	$X(\omega_{13}) = 10$
$S_4^1(\omega_{12}) = 40$	$\frac{140}{4} = 35$	$X(\omega_{12}) = 0$
$S_4^1(\omega_{11}) = 80$	$\frac{160}{4} = 40$	$X(\omega_{11}) = 5$
$S_4^1(\omega_{10}) = 40$	$\frac{120}{4} = 30$	$X(\omega_{10}) = 0$
$S_4^1(\omega_9) = 40$	$\frac{100}{4} = 25$	$X(\omega_9) = 0$
$S_4^1(\omega_8) = 20$	$\frac{80}{4} = 20$	$X(\omega_8) = 0$
$S_4^1(\omega_7) = 80$	$\frac{150}{4} = \frac{75}{2}$	$X(\omega_7) = \frac{5}{2}$
$S_4^1(\omega_6) = 40$	$\frac{110}{4} = \frac{55}{2}$	$X(\omega_6) = 0$
$S_4^1(\omega_5) = 40$	$\frac{90}{4} = \frac{45}{2}$	$X(\omega_5) = 0$
$S_4^1(\omega_4) = 20$	$\frac{70}{4} = \frac{35}{2}$	$X(\omega_4) = 0$
$S_4^1(\omega_3) = 40$	$\frac{80}{4} = 20$	$X(\omega_3) = 0$
$S_4^1(\omega_2) = 20$	$\frac{60}{4} = 15$	$X(\omega_2) = 0$
$S_4^1(\omega_1) = 20$	$\frac{50}{4} = \frac{25}{2}$	$X(\omega_1) = 0$
$S_4^1(\omega_0) = 10$	$\frac{40}{4} = 10$	$X(\omega_0) = 0$

$$q = \frac{\frac{5}{4}-1}{2-1} = \frac{1}{4}; \quad \Pi_t(X) = \frac{1}{1+r} E_Q\left[\Pi_{t+1}(X) | \mathcal{F}_t\right] = \frac{4}{5}\left(\frac{1}{4}\Pi_{t+1}(X)(up) + \frac{3}{4}\Pi_{t+1}(X)(down)\right)$$

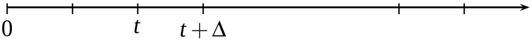
$$\Rightarrow \Pi_t(X) = \frac{1}{5}\Pi_{t+1}(X)(up) + \frac{3}{5}\Pi_{t+1}(X)(down)$$



4 Continuous Time Markets

- Aim: Introduce the model of Black, Merton and Scholes and its generalizations
- Why Continuous time models?
  - Assets are (in many cases) quoted at high frequency and without interruption; Ex: Foreign Exchange rates and Stock-indices

    $\Rightarrow$ Almost continuously quoted
  - A robust discrete time model must give predictions almost independent of the time increment Δ, when Δ is small:



    $\Rightarrow$ Continuous limit $\exists$

- Quotes are not equidistant in reality:

$\Rightarrow$ Quotes can be considered as a sample of a Continuous Time Process

- Technical reasons:
  Discrete mathematics complicated
  Theory of continuous time processes $\exists$ and computational easier (Stochastic integration, Itô's calculus, Girsanov's transformation, ...)

4.1 Original Black-Scholes Model

Stock price model

- Trading dates: $\mathbb{T} = [0, T]$
- Random source: A one dimensional Brownian motion W is defined on a (complete) probability space $(\Omega, P, \mathcal{F})$, where P is the a priori probability measure.
- Filtration: $\{\mathcal{F}_t\}_{t \in \mathbb{T}}$ generated by W (and the null-sets); $\mathcal{F} = \mathcal{F}_T$
- Two assets: A Risk-free bank account, with price process

$$S_t^0 = S_0^0 \exp(rt) \tag{41}$$

and a Risky stock with price process, which is an "exponential Brownian"

$$S_t^1 = S_0^1 \exp(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t), \tag{42}$$

where $r, \mu, \sigma \in \mathbb{R}$ and $\sigma \neq 0$. S_t^1 / S_0^1 is log-normal; $\ln(S_t^1 / S_0^1)$ is $\mathcal{N}((\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t)$

- SDEs (Stochastic Differential Equation) by Itô's lemma:

$$dS_t^0 = rS_t^0 dt \tag{43}$$
$$dS_t^1 = S_t^1 \mu dt + S_t^1 \sigma dW_t. \tag{44}$$

Here r is the (continuous compounded) spot interest rate, μ the drift and σ the volatility.

- Discounted price $\bar{S}$: $\bar{S}^i = S^i / S^0$

$$d\bar{S}_t^0 = 0 \tag{45}$$
$$d\bar{S}_t^1 = \bar{S}_t^1 (\mu - r) dt + \bar{S}_t^1 \sigma dW_t. \tag{46}$$

Exercise 4.1

Let $\gamma = \frac{\mu - r}{\sigma}$ and $\xi_t = \exp(-\gamma W_t - \frac{1}{2}\gamma^2 t)$. Establish that $\xi \bar{S}$ is a P-martingale. (N.b. γ is called the market price of risk).

Portfolio

- The portfolio $\theta = (\theta^0, \theta^1)$:
  - θ^0_t is the number of units of the risk-free asset held at time t
  - θ^1_t is the number of units of the risky asset held at time t
  - θ^0_t and θ^1_t are known at time t, i.e. they are $\mathcal{F}_t$-measurable.
  - $\theta_t = (\theta^0_t, \theta^1_t)$ is the instantaneous portfolio at t.
- Price process $V(\theta)$ of a portfolio θ:

$$V_t(\theta) = \theta_t \cdot S_t, \ t \in \mathbb{T}. \tag{47}$$

- Discounted Price process $\bar{V}(\theta)$ of a portfolio θ: (Discounted to date 0)

$$\bar{V}_t(\theta) = V_t(\theta)/S^0_t. \tag{48}$$

Obviously $\bar{V}_t(\theta) = \theta_t \cdot \bar{S}_t$.

• Gains process $G(\theta)$ of prtf. θ: Sum of the gains from date 0 up to date t

$$G_t(\theta) = \int_0^t \theta_s \cdot dS_s, \; t \in \mathbb{T}. \tag{49}$$

Warning: To be rigorous, one should here introduce a set of admissible portfolios, such that the gains process is well-defined and such that the model is arbitrage free (excluding for example doubling strategies). This is outside the scope of this course and we shall just suppose that the prtf. is sufficiently integrable or uniformly bounded from below, as to guarantee these properties.

• Discounted Gains process $\bar{G}(\theta)$ of prtf. θ:

$$\bar{G}_t(\theta) = \int_0^t \theta_s \cdot d\bar{S}_s, \; t \in \mathbb{T}. \tag{50}$$

• Self-financing prtf. θ is a prtf. where the changes in its price only comes from variations in the asset prices, i.e.

$$V_t(\theta) = V_0(\theta) + G_t(\theta), \quad \forall t \in \mathbb{T}. \tag{51}$$

Exercise 4.2

Establish that the definition of a self-financing prtf. by formula (51) is equivalent to

$$\bar{V}_t(\theta) = \bar{V}_0(\theta) + \bar{G}_t(\theta), \quad \forall t \in \mathbb{T}. \tag{52}$$

Black-Scholes Equation

- Simple EU derivative X, i.e.

$$X = f(S^1_T), \quad \text{for some } f$$

- Hedging prtf. θ of X:

$$V_T(\theta) = X, \quad \text{for some self-fin. prtf. } \theta$$

- To construct the hedging prtf. θ of X we try the ansatz: There is a function $F \in C^{1,2}([0, T[\times]0, \infty[\,)$ s.t.

$$V_t(\theta) = F(t, S^1_t), \ \forall t \in \mathbb{T} \tag{53}$$

and

$$F(T, x) = f(x), \ \forall x > 0. \tag{54}$$

Introduce: $F_1(t, x) = \partial F(t, x)/\partial t$, $F_2(t, x) = \partial F(t, x)/\partial x$ and $F_{22}(t, x) = \partial^2 F(t, x)/\partial x^2$.

Differentiation of the l.h.s. of (53) gives, since θ is self-fin.:

$$dV_t(\theta) = \theta_t \cdot dS_t = (rS_t^0\theta_t^0 + \mu S_t^1\theta_t^1)dt + \sigma S_t^1\theta_t^1 dW_t \\ = (rV_t(\theta) + (\mu - r)S_t^1\theta_t^1)dt + \sigma S_t^1\theta_t^1 dW_t \quad (55)$$

(53) and (55) give,

$$dV_t(\theta) = (rF(t, S_t^1) + (\mu - r)S_t^1\theta_t^1)dt + \sigma S_t^1\theta_t^1 dW_t \quad (56)$$

Differentiation of the r.h.s. of (53) gives, by Itô's lemma, $\forall t \in [0, T[$:

$$dF(t, S_t^1) = (F_1(t, S_t^1) + \mu S_t^1 F_2(t, S_t^1) + \frac{1}{2}\sigma^2 (S_t^1)^2 F_{22}(t, S_t^1))dt \\ + \sigma S_t^1 F_2(t, S_t^1) dW_t. \quad (57)$$

Identification of (56) and (57) gives, first

$$\theta_t^1 = F_2(t, S_t^1), \tag{58}$$

and then

$$rF(t, S_t^1)+(\mu-r)S_t^1\theta_t^1 = F_1(t, S_t^1)+\mu S_t^1 F_2(t, S_t^1)+\frac{1}{2}\sigma^2(S_t^1)^2 F_{22}(t, S_t^1). \tag{59}$$

- Eqs. (59) and (54) give the Black-Scholes Equation:

$\forall\,(t,x) \in \mathbb{T}\times]0,\infty[$,

$$\begin{cases} \frac{\partial}{\partial t}F(t,x) + rx\frac{\partial}{\partial x}F(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}F(t,x) = rF(t,x), \\ F(T,x) = f(x),\ \forall x > 0. \end{cases} \tag{60}$$

- The hedging prtf. θ is given by (58)

$$\begin{cases} \theta_t^1 = F_2(t, S_t^1) \\ \theta_t^0 = \frac{1}{S_t^0}\left(F(t, S_t^1) - F_2(t, S_t^1)S_t^1\right). \end{cases} \tag{61}$$

A special case of the Feynman-Kač formula gives the solution of the B-S eq.:

Theorem 4.3

(Under certain conditions on f). If

$$F(t,x) = e^{-r(T-t)} \int_{-\infty}^{\infty} f(xe^{y}) \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp\left(-\frac{(y-(T-t)(r-\frac{1}{2}\sigma^2))^2}{2\sigma^2(T-t)}\right) dy, \quad (62)$$

where $(t,x) \in [0,T[\, \times\,]0,\infty[$, then F is the solution of the B-S Eq. (60) and θ defined by (61) is a hedging prtf. of $X = f(S^1_T)$.

Proof: □

Black-Scholes Formula for a Call

- Let f be the pay-off of a EU Call with strike $K > 0$:

$$f(x) = (x - K)_+, \; x > 0 \tag{63}$$

and let C be the solution of the B-S eq. given by formula (62). Then

$$\begin{aligned} C(t,x) &= e^{-r(T-t)} \int_{-\infty}^{\infty} (xe^y - K)_+ \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp\left(-\frac{(y-(T-t)(r-\frac{1}{2}\sigma^2))^2}{2\sigma^2(T-t)}\right) dy \\ &= e^{-r(T-t)} \int_{\ln(K/x)}^{\infty} (xe^y - K) \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp\left(-\frac{(y-(T-t)(r-\frac{1}{2}\sigma^2))^2}{2\sigma^2(T-t)}\right) dy. \end{aligned} \tag{64}$$

We make the substitution

$$z = -\frac{y - (T-t)(r - \sigma^2/2)}{\sigma\sqrt{T-t}}, \;\; \text{so} \;\; y = -z\sigma\sqrt{T-t} + (T-t)(r - \sigma^2/2).$$

Let

$$z_0 = \frac{\ln(x/K) + (T-t)(r - \sigma^2/2)}{\sigma\sqrt{T-t}}.$$

Formula (64) now gives

$$C(t,x) = e^{-r(T-t)} \int_{-\infty}^{z_0} \left(x \exp(-z\sigma\sqrt{T-t} + (T-t)(r-\sigma^2/2)) - K \right) \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} d \tag{65}$$

Let

$$I_1(t,x) = e^{-r(T-t)} \int_{-\infty}^{z_0} x \exp(-z\sigma\sqrt{T-t} + (T-t)(r-\sigma^2/2)) \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz$$

and

$$I_2(t,x) = -Ke^{-r(T-t)} \int_{-\infty}^{z_0} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz.$$

Then

$$C(t,x) = I_1(t,x) + I_2(t,x).$$

Let N be the normal $(0,1)$ pdf (probability distribution function).

It follows that

$$I_1(t,x) = x\, e^{-r(T-t)} \int_{-\infty}^{z_0} \exp(-z\sigma\sqrt{T-t} + (T-t)(r-\sigma^2/2)) \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz$$
$$= x \int_{-\infty}^{z_0} \exp\left(-\frac{(z+\sigma\sqrt{T-t})^2}{2}\right) \frac{dz}{\sqrt{2\pi}} = x\, N(z_0 + \sigma\sqrt{T-t}\ ).$$

For I_2 it follows that

$$I_2(t,x) = -Ke^{-r(T-t)} N(z_0).$$

To sum up, introduce $\tau = T - t$, $d_1(\tau) = z_0 + \sigma\sqrt{\tau}$ and $d_2(\tau) = z_0$, i.e.

$$d_1(\tau) = \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{x}{K} + \tau(r + \frac{\sigma^2}{2})) \quad (66)$$
$$d_2(\tau) = \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{x}{K} + \tau(r - \frac{\sigma^2}{2})), \quad (67)$$

then (65) gives the Black-Scholes Formula for the price of a EU Call

$$C(t,x) = xN(d_1(\tau)) - Ke^{-r\tau} N(d_2(\tau)). \quad (68)$$

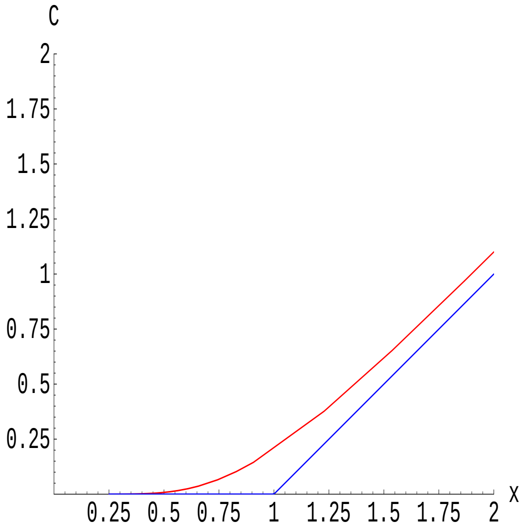


Figure: Price $C(0, x)$. Here $T = 1$, $K = 1$, $r = 1/10$ and $\sigma = 4/10$

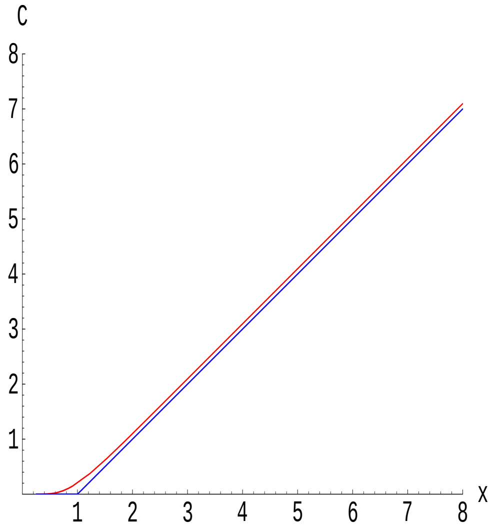


Figure: Price $C(0,x)$. Here $T=1$, $K=1$, $r=1/10$ and $\sigma=4/10$

- Call-Put relation: Let $P(T,x)=(K-x)_+$. Then

$$C(T,S_T^1)-P(T,S_T^1)=S_T^1-K.$$

The arbitrage price at time t of each side then gives

$$C(t,S_t^1)-P(t,S_t^1)=S_t^1-Ke^{-r(T-t)}. \qquad (69)$$

Since this is true for all values of S_t^1, we obtain the Call-Put relation

$$C(t,x)-P(t,x)=x-Ke^{-r(T-t)}, \qquad (70)$$

for $x>0$.

- Price of a Put $P(t,x)$ is obtained from formulas (68) and (70) and the relation $1-N(z)=N(-z)$

$$P(t,x)=-xN(-d_1(\tau))+Ke^{-r\tau}N(-d_2(\tau)). \qquad (71)$$

for $x>0$.

Example 4.4

Consider a standard B-S market, with annual interest rate $r = 3.5\%$ and annual volatility $\sigma = 30\%$. The stock price is 132€ today. What is the price today of a Call and a Put, with 9 months to maturity and strike 125?

Let t be today. We have $\tau = 3/4$Y and according to (66) and (67): $d_1(\tau) = \frac{1}{3/10\sqrt{3/4}}(\ln(132/125) + 3/4(35/1000 + (3/10)^2/2)) \approx 0.440665$ and $d_2(\tau) = \frac{1}{3/10\sqrt{3/4}}(\ln(132/125) + 3/4(35/1000 - (3/10)^2/2)) \approx 0.180858$. $N(d_1(\tau)) \approx 0.670272$ and $N(d_2(\tau)) \approx 0.57176$. Formula (68) gives $C(t, 132) \approx 18,8576$. The Call-Put parity relation (70) then gives $P(t, 132) \approx 8.61903$.

4.2 The greeks

• B-S model is complete, so by definition every derivative is hedgeable

• However, in practice a hedge is in general only approximate for several reasons, as

1) The portfolio can not be re-balanced continuously in time.

2) The underlying price model, here the original B-S model, is only an approximation of the real price dynamics.

3) The model parameters are only determined with a certain accuracy.

• Let F be the price function (here given by B-S eq. (60)) of a simple EU derivative X and let θ be an approximate hedging prtf., s.t. its value at time t is a function of the stock price S_t^1. Then the price difference p at t of derivative and hedge is:

$$p(t, S_t^1) = F(t, S_t^1) - V_t(\theta). \tag{72}$$

- p is also a function of the model parameters r, μ and σ, if this is the case for θ. Sensitivities of derivative price $F(t,x)$, w.r.t. variations in t, x, r and σ needed;
- "Greeks" for a Call $F(t,x) = C(t,x)$:

$$\text{Delta} \quad \Delta = \frac{\partial C(t,x)}{\partial x} = N(d_1) > 0, \quad \text{for} \quad x > 0. \tag{73}$$

$$\text{Gamma} \quad \Gamma = \frac{\partial^2 C(t,x)}{\partial x^2} = \frac{1}{x\sigma\sqrt{\tau}} N'(d_1) > 0, \quad \text{for} \quad x > 0. \tag{74}$$

(So, the Call-price is strictly concave.)

$$\text{Theta} \quad \Theta = \frac{\partial C}{\partial t} = -\frac{x\sigma}{2\sqrt{\tau}} N'(d_1) - Ke^{-r\tau} r\, N(d_2) < 0, \quad \text{for} \quad x > 0, \tag{75}$$

where $\tau = T - t$. (So C is increasing in τ.)

$$\text{Vega} \quad V = \frac{\partial C}{\partial \sigma} = x\sqrt{\tau}\, N'(d_1) > 0, \quad \text{for} \quad x > 0. \tag{76}$$

$$\text{Rho} \quad \rho = \frac{\partial C}{\partial r} = K\tau e^{-r\tau}\, N(d_2) > 0, \quad \text{for} \quad x > 0. \tag{77}$$

• Δ, Γ and Θ satisfies, for a simple derivative with price F (see B-S eq. (60)) :

$$\Theta + rx\Delta + \frac{1}{2}\sigma^2 x^2 \Gamma - rF = 0. \tag{78}$$

Example 4.5

In the situation of Example 4.4, what are today the hedging portfolios of the Call and the Put respectively?
The hedging prtfs. are obtained from (61). Let $x = 132$. In the case of the Call, eq. (73) gives

$$\theta_t^1 = N(d_1) \approx 0.670272 \quad \text{stocks}$$

and

$$\theta_t^0 S_t^0 = C(t, x) - N(d_1)x \approx -69.6184 \text{ €}.$$