Etude loi Binomiale

- $\mathcal{B}(n, p)$
- Support $D_x = \{0, 1 \ldots, n\}$
- $p_x = \begin{cases} C_n^x \, p^x (1-p)^{n-x} & si \; x \in D_x \\ 0 & ailleurs \end{cases}$

- $E[X] = \sum_{x \in \mathbb{Z}} x \, p_x(x) = \sum_{x \in D_x} C_n^x p^x (1-p)^{n-x} \cdot x$

$$= \sum_{x \in D_x} \frac{n!}{x!(n-x)!} \cdot p^x \cdot (1-p)^{n-x} \cdot x$$

$$= \sum_{1}^{n} (n \cdot p) \frac{(n-1)!}{(x-1)!(n-1-(x-1))!} \cdot p^{x-1} \cdot (1-p)^{n-x}$$

$$= \sum_{0}^{n-1} (n \cdot p) \cdot \frac{(n-1)!}{x!(n-1-x)!} \cdot p^x \cdot (1-p)^{n-1-x}$$

$$= n \cdot p \cdot (p + 1 - p)^{n-1}$$

$$= np$$

- $E[X^2] = \sum_{0}^{n} x^2 \, C_n^x \, p^x (1-p)^{n-x}$

$$= \sum_{0}^{n} x^2 \cdot \frac{n!}{x!(n-x)!} \cdot p^x \cdot (1-p)^{n-x}$$

$$= \sum_{1}^{n} x \cdot \frac{(n-1)!}{(x-1)!(n-x)!} \cdot p^{x-1} (1-p)^{n-x} \cdot np$$

$$= np \left(\sum_{0}^{n-1} (x+1) \left[\frac{(n-1)!}{x!(n-1-x)!} \cdot p^x \cdot (1-p)^{n-1-x} \right] \right)$$

$$= np \left(\sum_{0}^{n-1} x \cdot \frac{(n-1)!}{x!(n-1-x)!} \cdot p^x \cdot (1-p)^{n-1-x} + \sum_{0}^{n-1} \frac{(n-1)!}{x!(n-1-x)!} \cdot p^x \cdot (1-p)^{n-1-x} \right)$$

$$= np \left((n-1)p + 1 \right)$$

$$\Rightarrow \quad Var[X] = E[X^2] - E[X]^2$$

$$= np((n-1)p + 1) - (np)^2$$

$$= np(np - p + 1) - n^2p^2$$

$$= (np)^2 + np(1-p) - (np)^2$$

$$= np(1-p)$$

Etude loi de Poisson

- $P[\lambda]$

- Support $D_x : \mathbb{N}$

- $p_x = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!} & \text{si } x \in D_x \\ 0 & \text{ailleurs} \end{cases}$

- $E[X] = \sum x\, p_x(x) = \sum_0^{+\infty} x \cdot \dfrac{e^{-\lambda} \cdot \lambda^x}{x!}$

$$= \sum_1^{+\infty} \frac{e^{-\lambda} \cdot \lambda^x}{(x-1)!}$$

$$= \lambda \sum_0^{+\infty} \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$= \lambda$$

- $E[X^2] = \sum x^2 p_x(x) = \sum_0^{+\infty} x^2 \dfrac{e^{-\lambda} \cdot \lambda^x}{x!}$

$$= \sum_1^{+\infty} x \cdot \frac{e^{-\lambda} \lambda^x}{(x-1)!}$$

$$= \sum_0^{+\infty} (x+1) \cdot \frac{e^{-\lambda} \cdot \lambda^{x+1}}{x!}$$

$$= \lambda \left(\sum_0^{+\infty} \frac{e^{-\lambda} \cdot \lambda^x}{x!} + \sum_0^{+\infty} x \frac{e^{-\lambda} \cdot \lambda^x}{x!} \right)$$

$$= \lambda (1 + \lambda)$$

$$= \lambda^2 + \lambda$$

$\Rightarrow Var[X] = E[X^2] - E[X]^2 = \lambda$

Approximation de la loi binomiale par poisson.

Soient $\{p_1, p_2 \dots p_n\}$ une suite de réels tq $\forall i \quad i p_i = \lambda$.

$$\Rightarrow \lim_{n \to +\infty} \binom{n}{x} p_n^{x} q^{n-x}$$

$$\lim_{n \to +\infty} \frac{n!}{x!(n-x)!} p_n^{x} \cdot \frac{n^{x}}{n^{x}} q^{n-x}$$

$$\lim_{n \to +\infty} \frac{1}{x!} \cdot \frac{n.(n-1)\dots(n-x+1)}{n.n\dots.n} (n p_n)^{x} q^{n-x}$$

$$\lim_{n \to +\infty} \frac{1}{x!} \cdot \underbrace{\left[\frac{n}{n} \cdot \frac{n-1}{n} \dots \frac{(n-x+1)}{n} \right]}_{x \text{ éléments}} \lambda^{x} \left(1 - \frac{n p_n}{n}\right)^{n-x}$$

$$\lim_{n \to +\infty} \frac{1}{x!} \cdot \lambda^{x} \cdot \left[\frac{n}{n} \cdot \frac{(n-1)}{n} \dots \frac{n-x+1}{n} \right] \cdot \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

* $\lim_{n \to +\infty} \left[\frac{n}{n} \cdot \frac{(n-1)}{n} \dots \frac{n-x+1}{n} \right] = 1$

* $\lim_{n \to +\infty} \left(1 - \frac{\lambda}{n}\right)^{n-x} = e^{-\lambda}$

$$\Rightarrow \lim_{n \to +\infty} \binom{n}{x} p_n^{x} q_n^{n-x} = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

Etude loi Uniforme

. $\mathcal{U}([a;b])$

. $C_X = [a;b]$

. $$f_X = \begin{cases} \frac{1}{b-a} & \text{si } x \in C_X \\ 0 & \text{sinon} \end{cases}$$

. $$F_X = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0 & \text{si } x < a \\ \frac{x-a}{b-a} & \text{si } a \leq x \leq b \\ 1 & \text{sinon} \end{cases}$$

. $$M_X(t) = E[e^{Xt}] = \int_{-\infty}^{+\infty} e^{xt} \cdot f_X(t)\,dx$$

$$= \int_a^b e^{xt} \cdot \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a} \cdot \left[\frac{e^{xt}}{t} \right]_a^b \qquad t \neq 0$$

$$= \frac{e^{tb} - e^{ta}}{t(b-a)}$$

. si $t = 0 \Rightarrow 1$

$$M_X(t) = \begin{cases} \frac{e^{tb} - e^{ta}}{t(b-a)} & t \neq 0 \\ 1 & \text{sinon} \end{cases}$$

• $E[X] = \left.\dfrac{d\,M_X(t)}{dt}\right|_{t=0} = \dfrac{1}{(b-a)} \cdot \dfrac{\left(be^{tb} - ae^{ta}\right)t - \left(e^{tb} - e^{ta}\right)}{t^2}$

$$= \frac{1}{b-a} \left(\frac{\left(bte^{tb} - e^{tb}\right)}{t^2} + \frac{\left(e^{ta} - ate^{ta}\right)}{t^2} \right)$$

$$* \quad e^{tb} \underset{V(0)}{\sim} 1 + tb + \frac{(tb)^2}{2}$$

$$\Rightarrow \quad bte^{tb} - e^{tb} \underset{V(0)}{\sim} bt(1+tb) - 1 - tb - \frac{(tb)^2}{2}$$

$$\underset{V(0)}{\sim} \frac{t^2b^2}{2} - 1$$

$$\Rightarrow \quad e^{at} - ate^{at} \underset{V(0)}{\sim} 1 - \frac{t^2a^2}{2}$$

$$\Rightarrow \quad E[X] = \frac{1}{(b-a)} \; \frac{b^2 - a^2}{2} = \frac{a+b}{2}$$

• $E[X^2] = \left.\dfrac{d^2M_X(t)}{dt^2}\right|_{t=0} = \dfrac{1}{b-a} \cdot \dfrac{t \times \left[b^2te^{tb} + \cancel{be^{tb}} - \cancel{be^{tb}} - a^2te^{at}\right] - 2\left[bte^{tb} - e^{tb} + e^{at} - ate^{ta}\right]}{t^3}$

$$= \frac{1}{b-a} \cdot \frac{e^{tb}\left(b^2t^2 - 2bt + 2\right) - e^{ta}\left(a^2t^2 - 2at + 2\right)}{t^3}$$

$$* \quad e^{tb} \underset{V(0)}{\sim} 1 + tb + \frac{(tb)^2}{2} + \frac{(tb)^3}{3!} \Rightarrow e^{tb}\left(b^2t^2 - 2bt + 2\right) \underset{V(0)}{\sim} \cancel{b^2t^2} + b^3t^3 - \cancel{2tb} - 2t^3b^2 \ldots t^3b^3$$

$$+ 2 + \cancel{2tb} + t^2\cancel{b^2} + \frac{t^3b^3}{3}$$

$$\underset{V(0)}{\sim} \frac{t^3b^3}{3}$$

$$* \quad e^{ta} \underset{V(0)}{\sim} \frac{t^3b^3}{3}$$

$$\Rightarrow \quad E[X^2] = \frac{a^2 + ab + b^2}{3}$$

• $\mathrm{Var}[X] = \dfrac{(b-a)^2}{12}$

Loi exponentielle

. $Exp(\theta, \nu)$

. $C_x =]\nu ; +\infty[$

. $f_x = \begin{cases} \theta e^{-\theta(t-\nu)} & \text{si } x \in C_x \\ 0 & \text{ailleurs} \end{cases}$

. $F_x(x) = \begin{cases} 1 - e^{-\theta(t-\nu)} & \text{si } x \in C_x \\ 0 & \text{sinon} \end{cases}$

. $M_x(t) = E[e^{xt}] = \int_{-\infty}^{+\infty} e^{xt} \cdot f_x(x)\,dx$

$$= \int_{\nu}^{+\infty} e^{xt} \cdot \theta e^{-\theta(x-\nu)} \, dx$$

$$= \theta e^{\theta \nu} \int_{\nu}^{+\infty} e^{-x(\theta - t)} \, dx$$

$$= \theta e^{\theta \nu} \left[\frac{e^{-x(\theta - t)}}{\theta - t} \right]_{+\infty}^{\nu}$$

$$= \theta e^{\theta \nu} \cdot \frac{e^{-\nu\theta} \cdot e^{\nu t}}{\theta - t}$$

$$= \frac{\theta \cdot e^{\nu t}}{\theta - t}$$

- $$E[X] = \left.\frac{d M_X(t)}{dt}\right|_{t=0} = \frac{\theta v e^{tv}(\theta - t) + \theta e^{tv}}{(\theta - t)^2}$$

$$\underset{(0)}{=} \frac{\theta^2 v + \theta}{\theta^2} \underset{(0)}{=} v + \frac{1}{\theta}$$

- $$E[X^2] = \left.\frac{d^2 M_X(t)}{dt^2}\right|_{t=0} = \frac{e^{tv}(\theta^2 v^2 - \theta v^2 t + \cancel{\theta v} - \cancel{\theta v})(\theta - t) + 2e^{tv}(\theta^2 v - \theta v t + \theta)}{(\theta - t)^3}$$

$$\underset{(0)}{=} \frac{\theta^3 v^2 + 2\theta^2 v + 2\theta}{\theta^3}$$

$$\Rightarrow Var[X] = E[X^2] - E[X]^2$$

$$= \left(v^2 + \frac{1}{\theta^2} + \frac{v}{\theta}\right) - \left(v + \frac{1}{\theta}\right)^2$$

$$= v^2 + \frac{2}{\theta^2} + \frac{2v}{\theta} - v^2 - \frac{1}{\theta^2} - 2\frac{v}{\theta}$$

$$= \frac{1}{\theta^2}$$

Loi Normale

$\mathcal{N}(\mu, \sigma^2)$

$C_X = \mathbb{R}$

. $f_X(x) = \dfrac{1}{\sigma\sqrt{2\pi}} . e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

. $F_X(x) = \displaystyle\int_{-\infty}^{x} f_X(t)\, dt$

. $M_X(t) = E[e^{xt}] = \displaystyle\int_{-\infty}^{+\infty} e^{xt} . \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

$= \dfrac{1}{\sigma . \sqrt{2\pi}} \displaystyle\int_{-\infty}^{+\infty} e^{xt} e^{-\left(\frac{x-\mu}{\sigma}\right)^2 . \frac{1}{2}} dx$

On pose $X = \dfrac{x-\mu}{\sigma} \Rightarrow dX = \dfrac{dx}{\sigma}$

$\Rightarrow M_X(t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{+\infty} e^{(\sigma X + \mu)t - \frac{X^2}{2}} dX$

$= \dfrac{e^{\mu t}}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{+\infty} e^{\sigma t X - \frac{X^2}{2} + \frac{(\sigma t)^2}{2} - \frac{(\sigma t)^2}{2}} dX$

$= \dfrac{e^{\mu t}}{\sqrt{2\pi}} . e^{\frac{(\sigma t)^2}{2}} \displaystyle\int_{-\infty}^{+\infty} e^{-\frac{(X - \sigma t)^2}{2}} dX$

$= e^{t\mu + \frac{(\sigma t)^2}{2}}$

$\bullet\ E[X] = \left.\dfrac{d M_X(t)}{dt}\right|_{t=0} = \mu$

$\bullet\ E[X^2] = \dfrac{d M_X(t)}{dt} = \left(e^{\mu t + \frac{\sigma^2 t^2}{2}}\right)''$

$= \left((\mu + t\sigma^2) e^{\mu t + \frac{\sigma^2 t^2}{2}}\right)'$

$= e^{\mu t + \frac{\sigma^2 t^2}{2}} \left(\sigma^2 + [\mu^2 + t\sigma^2\mu + t\sigma^2\mu + t^2\sigma^2]\right.$

$= \sigma^2 + \mu^2$

$\Rightarrow Var[X] = \sigma^2$

Loi Lognormale

<u>Rappel</u> : • X v.a.c. de support C_X.

• $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ monotone telle que

$$Y = \phi(X)$$

• On note $\psi = \phi^{-1}$

Alors Y est v.a.c de support $C_Y = \phi(C_X)$ et de fct de densité :

$$f_Y(y) = \begin{cases} f_X(\psi(y)) . |\psi'(y)| & \text{si } y \in C_Y \\ 0 & \text{sinon} \end{cases}$$

On pose $\begin{cases} \varphi(x) = e^x \; ; \; \varphi'(x) = e^x > 0 \text{ monotone} \\ \psi(\cancel{y}) = \ln \cancel{y} \; ; \; \psi'(\cancel{y}) = \dfrac{1}{\cancel{y}} \end{cases}$

• $C_Y =]0 ; +\infty[$

• $$f_Y(y) = \begin{cases} \dfrac{1}{y} . \dfrac{1}{\sigma\sqrt{2\pi}} . e^{-\frac{(\ln y - \mu)^2}{\sigma^2 . 2}} & \text{si } y > 0 \\ 0 & \text{sinon} \end{cases}$$

- $F_Y(y) = \int_{-\infty}^{y} f_Y(x)\, dx = \int_{0}^{y} \frac{1}{x\sigma\sqrt{2\pi}} \cdot e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} dx$

$X = \frac{\ln x - \mu}{\sigma} \Rightarrow dx = \sigma x \, dX$

$\Rightarrow F_Y(y) = \int_{-\infty}^{\frac{\ln y - \mu}{\sigma}} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{X^2}{2}} dX$

$= F_{\mathcal{N}(0,1)}\left(\frac{\ln y - \mu}{\sigma}\right)$

avec $F_{\mathcal{N}(0,1)}$ la fct° de répartition de la loi normale centrée réduite.

- On a $X : \mathcal{N}(\mu_x, \sigma_x^2) \Rightarrow M_X(t) = e^{\mu_x t + \frac{\sigma_x^2 t^2}{2}}$

$\rightsquigarrow E[Y] = E[e^{\ln Y}] = E[e^X] = M_X(1)$

$\Rightarrow E[Y] = e^{\mu_x + \frac{\sigma_x^2}{2}}$

$\rightsquigarrow E[Y^2] = E[e^{2\ln Y}] = E[e^{2X}] = M_X(2)$

$\Rightarrow E[Y^2] = e^{2\mu_x + 2\sigma_x^2}$

$\rightsquigarrow Var[Y] = E[Y^2] - E[Y]^2$

$= e^{2\mu_x} e^{2\sigma_x^2} - e^{2\mu_x} e^{\sigma_x^2}$

$= e^{2\mu_x + \sigma_x^2}\left(e^{\sigma_x^2} - 1\right)$