Formulaire Probas TI

1/ Bases

Fonction de répartition : $F_X(x) = P(X \leq x)$ densité : dériver répart°

Fonction de masse ~~ou densité~~ : $p_X(x) = P(X = x)$

Supports variable aléatoire :

- discrète : $D_x = \{x : p_X(x) > 0\}$, $\sum_{x \in D_x} p_X(x) = 1$
- continue : $C_x = \{x : f_X(x) > 0\}$, $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

$(F'_X(x) = f_X(x))$

Espérance :

- discret : $E[X] = \sum_{x \in D_x} x \cdot p_X(x)$
- continu : $E[X] = \mu_X = \int_{-\infty}^{+\infty} x \cdot f_X(x)\,dx$

k^{e} moment d'une var. alé. continue

- origine : $\mu'_k = E[X^k]$
- p.r. moyenne : $\mu_k = E[(X - \mu_X)^k]$

Variance : $\sigma_X^2 = \mu_2 = E[(X - E[X])^2] = E[X^2] - E[X]^2$

Ecart-type : $\sigma_X = \sqrt{\sigma_X^2}$

Fct génératrice des moments :

$$M_X(t) = E[e^{Xt}] = \int_{-\infty}^{+\infty} e^{xt} \cdot f_X(x)\,dx$$

Fct carac. ~~des moments~~ :

$$\Phi_X = E[e^{i\omega X}] = \int_{-\infty}^{+\infty} e^{i\omega x} \cdot f_X(x)\,dx$$

$$C_n^k = \frac{n!}{k!(n-k)!}$$

TCL :

$X_1, \dots, X_n$ v.a. idpts. centrées, de variance σ_i^2

$S_n = \sum_{i=1}^{n} X_i$; $s_n^2 = \sum_{i=1}^{n} \sigma_i^2$; $F_i(x)$ fct de réparts.

$$\forall \varepsilon > 0, \lim_{n \to +\infty} \left(\frac{1}{s_n^2} \sum_{i=1}^{n} \int_{|x_i| > \varepsilon s_n} x_i^2 \, dF_i(x) \right) = 0$$

$$\Rightarrow \frac{S_n}{s_n} \xrightarrow{n \to +\infty} \mathcal{N}(0,1)$$

I. Lois de probas

Loi binomiale : $X \sim \mathcal{B}(n,p)$

(support) $D_x = \{0; \dots; n\}$

(masse) $P_X(x) = \begin{cases} C_n^x p^x (1-p)^{n-x}, & \forall x \in D_x \\ 0 & \text{sinon} \end{cases}$

(espérance) $E[X] = np$ $\quad M_X(t) = [(1-p) + pe^t]^n$

(variance) $\sigma_X^2 = np(1-p)$ $\quad \Phi_X(t) = [(1-p) + pe^{it}]^n$

Loi Bernoulli : $X \sim B(p)$ = loi binomiale avec $n = 1$

Loi Poisson : $X \sim P[\lambda]$

$D_x = \{0; \dots; n\}$ $\quad E[X] = \lambda$ $\quad M_X(t) = \exp(\lambda(e^t - 1))$

$P_X(x \in D_x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$ $\quad \sigma_X^2 = \lambda$ $\quad \Phi_X(t) = \exp(\lambda(e^{it} - 1))$

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Loi uniforme : $X \sim \mathcal{U}([a;b])$

(support) $C_x = [a;b]$

(densité) $f_X(x \in C_x) = \dfrac{1}{b-a}$

(répartition) $F_X(x \in \underset{(\text{donc } C_x)}{[a;b]}) = \dfrac{x-a}{b-a}$ (0 si $x < a$, 1 si $x > b$)

(génératrice) $M_X(t) = \begin{cases} \dfrac{e^{tb} - e^{ta}}{t(b-a)} & ; x \in C_x \\ 1 & ; t = 0 \end{cases}$ $\quad E[X] = \dfrac{a+b}{2}$ $\quad \sigma_X^2 = \dfrac{(b-a)^2}{12}$

(caractéristique) $\Phi_X(\omega) = \dfrac{e^{i\omega b} - e^{i\omega a}}{i\omega(b-a)}$

Loi normale : $X \sim \mathcal{N}(\mu, \sigma^2)$ ($\mu = 0$, $\sigma^2 = 1$ $\Leftrightarrow$ var. aléa. centrée réduite)

$C_x = \mathbb{R}$

$f_X(x \in C_x) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2}\right)$

$F_X(x \in C_x) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\dfrac{1}{2}t^2\right) dt$

$M_X(t) = \exp\left(\mu t + \dfrac{\sigma^2 t^2}{2}\right)$

$\Phi_X(t) = \exp\left(\mu i t - \dfrac{\sigma^2 t^2}{2}\right)$

$E[X] = \mu$

$\sigma_X^2 = \sigma^2$

Loi exponentielle :

$C_x = [0; +\infty[$ $\quad E[X] = \dfrac{1}{\lambda}$

$f_X(x) = \lambda e^{-\lambda x}$ $\quad \sigma_X^2 = \dfrac{1}{\lambda^2}$

$F_X(x) = 1 - e^{-\lambda x}$

$M_X(t) = \left(1 - \dfrac{t}{\lambda}\right)^{-1}$

$\Phi_X(t) = \left(1 - \dfrac{it}{\lambda}\right)^{-1}$

Mémo :

$$C_n^x = \frac{n!}{x!(n-x)!}$$

Bino : $X \sim \mathcal{B}(n,p)$	$D_x = \{0 ; \dots ; n\}$ $P_X(x) = C_n^x p^x (1-p)^{n-x}$	$E[X] = np$ $\sigma_X^2 = np(1-p)$
Poisson $X \sim \mathcal{P}[\lambda]$	$D_x = \{0 ; \dots ; n\}$ $P_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$	$E[X] = \lambda$ $\sigma_X^2 = \lambda$
Uniforme $X \sim \mathcal{U}([a,b])$	$C_x = [a,b]$ $F_X(x) = \frac{x-a}{b-a}$	$E[X] = \frac{a+b}{2}$ $\sigma_X^2 = \frac{(b-a)^2}{12}$
Normale $X \sim \mathcal{N}(\mu,\sigma^2)$	$C_x = \mathbb{R}$ $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$E[X] = \mu$ $\sigma_X^2 = \sigma^2$ just in case
Exponentielle $X \sim \mathcal{E}[\lambda]$	$C_x = [0 ; +\infty[$ $F_X(x) = 1 - \exp(-\lambda x)$	$E[X] = 1/\lambda$ $\sigma_X^2 = 1/\lambda^2$
Normale centrée réduite $X \sim \mathcal{N}(0,1)$	$C_x = \mathbb{R}$	$E[X] = 0$ $\sigma_X^2 = 1$ $f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$ $F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{x^2}{2}\right) dx$ just in case

Espérance :

- discret : $E[X] = \sum_{x \in D_x} x \cdot P_X(x)$
- continu : $E[X] = \int_{-\infty}^{+\infty} x \cdot f_X(x)\, dx$

Variance : $\sigma_X^2 = E[X^2] - E[X]^2$

Génératrice des moments : $M_X(t) = E\left[e^{Xt}\right] = \int_{-\infty}^{+\infty} e^{xt} f_X(x)\, dx$

Caractéristique : $\Phi_X(\omega) = E\left[e^{i\omega X}\right] = \int_{-\infty}^{+\infty} e^{i\omega x} f_X(x)\, dx$

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

$$Y = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$$

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$$Y \sim \mathcal{N}(\mu, \sigma^2)$$
$$\Rightarrow Z = \frac{Y-\mu}{\sigma} \sim \mathcal{N}(0, 1)$$
$$\Rightarrow Y = \sigma Z + \mu$$

I. a. Fonction de répartition :

$$F_X(x) = P(X \leqslant x)$$
$$= P(\exp(Y) \leqslant x)$$
$$= P(Y \leqslant \ln x) \; ; \; x > 0$$
$$= P(\sigma Z + \mu \leqslant \ln x)$$
$$= P\left(Z \leqslant \frac{\ln x - \mu}{\sigma}\right)$$
$$= F_Z\left(\frac{\ln x - \mu}{\sigma}\right) \; ; \; \left(F_Z(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left(-\frac{x^2}{2}\right) dx\right)$$

$$f_X(x) = \frac{1}{\sigma x} f_Z\left(\frac{\ln x - \mu}{\sigma}\right) \left(f_Z(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\right)$$

II. i. $D_x = \{0; \dots; n\}$

$$C_n^x = \frac{n!}{x!(n-x)!}$$

$$P_X(x) = C_n^x p^x (1-p)^{n-x}$$
$$M_X(t) = E[e^{Xt}] = \sum_{i=0}^{n} e^{it} P_X(i) = \sum_{i=0}^{n} e^{it} \frac{n!\, p^i (1-p)^{n-i}}{i!(n-i)!}$$
$$= \sum_{i=0}^{n} e^{it} C_n^i p^i (1-p)^{n-i}$$

binôme de Newton $(a+b)^n = \sum_{i=0}^{n} C_n^i (\dots)$

$$= \left[(1-p) + pe^t\right]^n$$

ii. $D_x = \{0; \dots; n\}$

$$P_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$M_X(t) = E[e^{Xt}] = \sum_{i=0}^{n} e^{it} \frac{e^{-\lambda}\lambda^i}{i!} \quad \exp$$
$$= e^{-\lambda} \sum_{i=0}^{n} \frac{(\lambda e^t)^i}{i!}$$
$$= \exp(\lambda e^t - \lambda)$$
$$= \exp(\lambda(e^t - 1))$$

iii. loi binomiale $\mathcal{B}(180, 1/6)$

Appr. Poisson $\mathcal{P}\left(\frac{180}{6}\right) = \mathcal{P}(30)$

Normale $\mathcal{N}(\mu, \sigma^2)$

$$\mu = E[X] = 180/6 = 30$$
$$\sigma^2 = E[X^2] - E[X]^2 =$$

$\mathcal{B}$ $p_X(x) = C_n^x p^x (1-p)^{n-x}$

$\mathcal{P}$ $p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$

$\mathcal{U}$ $F_X(x) = \frac{x-a}{b-a}$ $f_X(x) = \frac{1}{b-a}$

$\mathcal{N}(0,1) : f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$

$\exp(\lambda) : F_X(x) = 1 - \exp(-\lambda x)$

$M_X(t) = E[e^{Xt}] = \int_{-\infty}^{+\infty} e^{xt} f_X(x)\,dx$

$X \sim \mathcal{U}([0;1])$

1. $D_{Y_n} = \{0; n\}$

$f_X(x) = 1$

$F_X(x) = x$

$P[Y_n = n] = P[X \in [0; \frac{1}{n}]]$
$= P[X \leq 1/n]$
$= F_X(1/n) = 1/n$

2. $P[Y_n = 0] = 1 - 1/n$

$E[Y_n] = 0 \cdot (1 - \frac{1}{n}) + n \cdot 1/n = 1$

$\sigma^2_{Y_n} = n - 1$

$E[Y_n^2] = 0 \cdot (1 - \frac{1}{n}) + n^2 \cdot 1/n = n$

$M_{Y_n}(t) = E[e^{Y_n t}] = e^{0t}(1 - \frac{1}{n}) + e^{nt} \cdot \frac{1}{n}$

$= 1 - \frac{1}{n} + \frac{e^{nt}}{n} = 1 + \frac{1}{n}(e^{nt} - 1)$

3. $\underset{n \to +\infty}{E[Y_n]} = 1$

$\underset{n \to +\infty}{M_{Y_n}(t)} = +\infty$ ($\cancel{A} = o(e^{nt})$

$\underset{n \to +\infty}{\sigma^2_{Y_n}} = +\infty$

ne cvge pas $\Rightarrow$ pas en loi

4. $\underset{n \to +\infty}{E[Y_n^2]} = +\infty \Rightarrow$ NOPE

$X \sim \ln \mathcal{N}(\mu, \sigma^2) \iff \ln X = Y \sim \underbrace{\mathcal{N}(\mu, \sigma^2)}_{\text{loi normale}}$

a. fct densité, fct répart° : X (termes de répar° $\mathcal{N}(0,1)$)

$Z \sim \mathcal{N}(0,1)$. $\forall x \in D_x$,

répart° $F_X(x) = \mathbb{P}(X \leq x)$] ln
$= \mathbb{P}(\underbrace{\ln X}_{Y} \leq \ln x)$
$= \mathbb{P}\left(\underbrace{\frac{Y-\mu}{\sigma}}_{} \leq \frac{\ln x - \mu}{\sigma}\right)$ $\frac{-\mu}{\sigma}$
$= \mathbb{P}\left(Z \leq \frac{\ln x - \mu}{\sigma}\right)$
$= F_Z\left(\frac{\ln x - \mu}{\sigma}\right)$

$\mathcal{N}(0,1)$: $\frac{\cancel{X}-\mu}{\sigma}$

densité : $f_X(x) = \frac{d}{dx} F_X(x) = f_Z\left(\frac{\ln x - \mu}{\sigma}\right) \cdot \frac{1}{\sigma x}$

$[Z \sim \mathcal{N}(0,1)] = \frac{1}{\sigma x} \exp\left(-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right)$

$f_Z(X) = \exp\left(-\frac{1}{2}X^2\right)$

$\mu = 2$
$\sigma = 1$
$P(X \leq 12) = F_X(12)$
$= F_Z(0{,}485)$

$M_Y(t) = \exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$

$\boxed{M_Y(t) = E(e^{t\cancel{X}}) \quad \text{(Fct génératrice des moments)}}$

$m = E(X) = E(e^Y)$
$= M_Y(1)$
$= \exp\left(\mu + \frac{\sigma^2}{2}\right)$

$X \sim \mathcal{B}(n,p)$ supp : $D_X = \{0, \dots, n\}$

masse : $p_X(k) = P(X=k) = C_n^k p^k (1-p)^{n-k}$

géné : $M_X(t) = E(e^{tX}) = [(1-p) + pe^t]^n$

défaur mules !

III) a. $f(x,y) = \begin{cases} Cxy & \forall (x,y),\ 0 \leq y \leq x \leq 1 \\ 0 & \text{sinon} \end{cases}$

$f_{X,Y}$ fct densité $\Leftrightarrow \iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy = 1$

$\Rightarrow \int_0^1 \int_0^x Cxy\,dx\,dy = \int_0^1 Cx \left[\int_0^x y\,dy\right] dx$

$= \int_0^1 \frac{1}{2} Cx \cdot x^2\,dx = \frac{C}{2} \int_0^1 x^3\,dx = \frac{C}{8} \Rightarrow C = 8.$

fcts de densité marginales :

$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y)\,dy = 8x \int_0^x y\,dy = 4x^3$

$f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x,y)\,dy = 8y \int_0^1 x\,dx = 4y$

Indépce ? Vérifier

$f_X(x) f_Y(y) = f_{X,Y}(x,y)$. Or, $f_X(x) f_Y(y) = 16x^3 y \neq 8xy$

$P_X(x) = C_n^x p^x (1-p)^{n-x}$

$P_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$

$F_X(x) = \frac{x-a}{b-a}$

$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$

$F_X(x) = 1 - \exp(-\lambda x)$

I/ $n \in \mathbb{N}^*$: $Y_n = \begin{cases} n & ; 0 \leqslant X \leqslant 1/n \\ 0 & ; X < 0 \text{ ou } X > \frac{1}{n} \end{cases}$; $X \sim \mathcal{U}([0;1])$

1. $P_{Y_n}(n) = P(Y_n = n)$

$D_{Y_n} = \{0; n\}$

$= P\left(X \in [0; \frac{1}{n}]\right) = \int_0^{1/n} dx = \frac{1}{n}$

$P(Y_n = 0) = 1 - \frac{1}{n}$

2. $E[X_n] = n \cdot P(Y_n = n) + 0 \cdot P(Y_n = 0) = 1$

$\sigma^2_{Y_n} = E[Y_n^2] - E[Y_n]^2 = n - 1$

$M_{Y_n}(t) = E(e^{tY_n}) = e^{nt} P(Y_n = n) + e^{0t} P(Y_n = 0)$

$= \frac{e^{nt}}{n} + 1 - \frac{1}{n}$

II/ $Y = \exp(X)$, $X \sim \mathcal{N}(\mu, \sigma^2)$

$F_Y(x) = P(Y \leq x) = P(\exp(X) \leq x) = \begin{cases} P(X \leqslant \ln x) = f_X(\ln x) \\ 0 \text{ si } x \leq 0 \end{cases}$

$X \sim \mathcal{N}(\mu, \sigma^2)$

$\Leftrightarrow Z = \frac{X - \mu}{\sigma} \sim \mathcal{N}(0,1) \Rightarrow X = \sigma Z + \mu$

$F_Y(x) = P(\sigma Z + \mu \leqslant \ln x) = P\left(Z \leqslant \frac{\ln x - \mu}{\sigma}\right)$

$f_Y(x) = F_Y'(x) = \frac{1}{\sigma y} f_Z\left(\frac{1}{\sigma}(\ln y - \mu)\right)$

$= \frac{1}{\sqrt{2\pi}\,\sigma y} \exp\left(\frac{-1}{2\sigma^2}(\ln y - \mu^2)\right)$

$E(Y) = E(\exp(X)) = E(\exp(\sigma Z + \mu)) = E(e^{\mu} \cdot \exp(\sigma Z)) = e^{\mu} E(\exp(\sigma Z))$

$E(e^{\sigma Z}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\sigma z - \frac{1}{2} z^2\right) dz$

$E(e^{\sigma Z}) = \int_{-\infty}^{+\infty} e^{\sigma z} \cdot \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \cdot dz$