Corrigé Probas - Mai 2012 (suite)

I a) $F_X(x) = \begin{cases} 0 & \text{si } x \leq 0 \\ F_Z\left(\frac{\ln x - \mu}{\sigma}\right) & \text{si } x > 0 \end{cases}$ où Z suit $\mathcal{N}(0,1)$

(cf Corrigé 2011)

$\Rightarrow$ Par dérivation $f_X(x) = \begin{cases} 0 & \text{si } x < 0 \\ \frac{1}{\sigma x} f_Z\left(\frac{\ln x - \mu}{\sigma}\right) & \text{si } x > 0 \end{cases}$

b) $P(X \leq 12) = F_X(12) = F_Z\left(\frac{\ln 12 - 2}{1}\right) = F_Z(2{,}485 - 2) = F_Z(0{,}485)$

$= \underline{68{,}6\,\%}$

c) $M_Y(t) = E(e^{tY}) = \exp\left(\mu t + t^2 \frac{\sigma^2}{2}\right)$.

$m = E(X) = E(e^Y) = M_Y(1) = \exp\left(\mu + \frac{\sigma^2}{2}\right)$

$$\Rightarrow \boxed{m = \exp\left(\mu + \frac{\sigma^2}{2}\right)}$$

II i) X suit $\mathcal{B}(n,p)$

$D_X = \{0, 1, 2, \dots, n\}$

$$P_X(x) = \begin{cases} C_n^x \, p^x (1-p)^{n-x} & \text{si } x \in D_X \\ 0 & \text{ailleurs} \end{cases}$$

$$M_X(t) = e^{tX} = \sum_{k=0}^{n} C_n^k \underbrace{p^k e^{tk}}_{(pe^t)^k} (1-p)^{n-k} = \left(pe^t + (1-p)\right)^n$$

d'après la formule du binôme de Newton

ii) Y suit $\mathcal{P}(\lambda)$

$$D_Y = \{0, 1, \dots\} = \mathbb{N}$$

$$P_Y(x) = \begin{cases} e^{-\lambda} \dfrac{\lambda^x}{x!} & \text{si } x \in D_Y \\ 0 & \text{ailleurs} \end{cases}$$

$$M_Y(t) = E(e^{tY}) = \sum_{k=0}^{\infty} \frac{e^{tk} e^{-\lambda} \lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!}$$

$$\Rightarrow \underline{M_Y(t) = e^{-\lambda} \cdot \exp(\lambda e^t)}$$

iii) Z suit $\mathcal{B}(180 ; \frac{1}{6})$

on peut approcher cette loi par $\mathcal{P}\left(\frac{180}{6}\right) = \mathcal{P}(30)$

ou par la loi normale $\mathcal{N}(\underset{\underset{\mu}{\uparrow}}{30} ; \underset{\underset{\sigma^2}{\uparrow}}{25})$

$\mathcal{B}(180 \, ; \, \frac{1}{6})$:

$$P(29 \leq Z \leq 32) = \sum_{k=29}^{32} C_{180}^{k} \, \frac{1}{6^k} \left(\frac{5}{6}\right)^{180-k} = \sum_{k=29}^{32} C_{180}^{k} \, \frac{5^{180-k}}{6^{180}}$$

$$P(31 \leq Z \leq 35) = \sum_{k=31}^{35} C_{180}^{k} \, \frac{5^{180-k}}{6^{180}}$$

$\mathcal{P}(30)$:

$$P(29 \leq Z \leq 32) = e^{-30} \sum_{k=29}^{32} \frac{180^k}{k!}$$

$$P(31 \leq Z \leq 35) = e^{-30} \sum_{k=31}^{35} \frac{180^k}{k!}$$

$\mathcal{N}(30 ; 25)$ $\quad U = \frac{Z - 30}{5}$ suit $\mathcal{N}(0,1)$

$$P(29 \leq Z \leq 32) = P\left(-\frac{1}{5} \leq U \leq \frac{2}{5}\right) = P\left(U \leq \frac{2}{5}\right) - P\left(U \leq -\frac{1}{5}\right)$$

$$P(29 \leq Z \leq 32) = P(U \leq 0{,}4) - 1 + P(U \leq 0{,}2)$$

$$= 0{,}6554 + 0{,}5793 - 1 = 1{,}2347 - 1 = 0{,}2347$$

$$= \underline{23{,}47\%}$$

$$P(31 \leq Z \leq 35) = P\left(\frac{1}{5} \leq U \leq 1\right) = P(U \leq 1) - P(0{,}2)$$

$$= 0{,}8413 - 0{,}5793 = 0{,}262 = \underline{26{,}2\%}$$