Corrigé Probas 2011

I $\quad Y = \ln X$ suit $\mathcal{N}(\mu, \sigma^2)$, $\mu = 5{,}38$, $\sigma = 10^{-2}$.

$$E(X) = E(e^Y) = 218 \text{ m}^3/\text{s}.$$

$$V(X) = V(e^Y) = (21{,}8)^2$$

1) $F_X(x) = P(X \leq x) = 0$ si $x \leq 0$

$= P(Y \leq \ln x)$ si $x > 0$

Y suit $\mathcal{N}(5{,}38\,;\,10^{-2}) \Rightarrow Z = \dfrac{Y-\mu}{\sigma} = \dfrac{Y - 5{,}38}{10^{-1}} = 10(Y - 5{,}38)$

suit $\mathcal{N}(0,1)$

$\Rightarrow$ Si $x > 0$ $\quad F_X(x) = P(Z \leq 10(\ln x - 5{,}38)) = F_Z(10(\ln x - 5{,}38))$

$$\boxed{F_X(x) = \begin{cases} 0 & \text{si } x \leq 0 \\ F_Z\left(10(\ln x - 5{,}38)\right) & \text{si } x > 0 \end{cases} \quad \text{où } Z \text{ suit } \mathcal{N}(0,1)}$$

2. $P(X > 200) = 1 - P(X \leq 200) = 1 - F_X(200) = 1 - F_Z\left(10(\ln 200 - 5{,}38)\right)$

$= 1 - F_Z\left(10(5{,}3 - 5{,}38)\right) = 1 - F_Z(-0{,}8) = 1 - \underbrace{P(Z \geq 0{,}8)}_{1 - F_Z(0{,}8)}$

$\Rightarrow P(X > 200) = F_Z(0{,}8) = 0{,}7881 = \boxed{78{,}81\%}$

3. $P(X > x) = 1 - F_Z\left(10(\ln x - 5{,}38)\right) = 0{,}72 = F_Z\left(10(5{,}38 - \ln x)\right)$

$\Rightarrow 10(5{,}38 - \ln x) = 0{,}58$

$\Rightarrow 5{,}38 - \ln x = 0{,}058 \quad \Rightarrow \ln x = 5{,}322$

$\Rightarrow \boxed{x \simeq 204{,}8 \ m^3/s}$

4) $W = X^2$

a) $F_W(w) = 0$ si $w \leq 0$

Si $w > 0$ $F_W(w) = P(W \leq w) = P(X \leq \sqrt{w})$

$= F_Z\left(10\left(\frac{1}{2}\ln w - 5{,}38\right)\right)$

$$\Rightarrow \boxed{F_W(w) = \begin{cases} 0 & \text{si } w \leq 0 \\ F_Z\left(10\left(\frac{1}{2}\ln w - 5{,}38\right)\right) & \text{si } w > 0 \end{cases}}$$

b) $f_W(w) = F'_W(w) = \begin{cases} 0 & \text{si } w \leq 0 \\ \frac{5}{w} f_Z(5 \ln w - 53{,}8) & \text{si } w > 0 \end{cases}$

avec $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$

c) $E(W) = E(X^2)$

$V(X) = E((X - E(X))^2) = E(X^2) - E(X)^2$

$\Rightarrow E(X^2) = V(X) + E(X)^2$

$\Rightarrow \boxed{E(W) = (21,8)^2 + (218,1)^2}$

II

1) $P_{ij}(t,s) = P(X_{t+s} = j \mid X_t = i)$ où $i, j \in E$

$E = \{A, B, C\} = \{1, 2, 3\}$

ne dépend que de t

2) Matrice de transition

$P_{ij} = P(X_{n+1} = j \mid X_n = i) = P(X_1 = j \mid X_0 = i)$

$$P_{11} = P(X_1 = 1 \mid X_0 = 1) = 0{,}1$$

$$P_{12} = P(X_1 = 2 \mid X_0 = 1) = 0{,}9$$

$$P_{11} + P_{12} + P_{13} = 1 \Rightarrow P_{13} = 0$$

$$P_{21} = P(X_1 = 1 \mid X_0 = 2) = 0{,}1$$

$$P_{23} = P(X_1 = 3 \mid X_0 = 2) = 0{,}4$$

$$P_{21} + P_{22} + P_{23} = 1 \Rightarrow P_{22} = 0{,}5$$

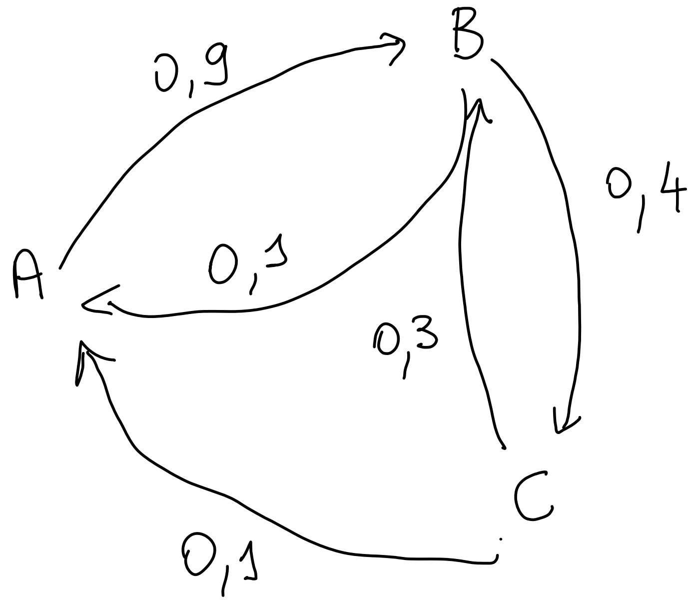
$$P_{31} = P(X_1 = 1 \mid X_0 = 3) = 0{,}1$$

$$P_{32} = P(X_1 = 2 \mid X_0 = 3) = 0{,}3$$

$$P_{31} + P_{32} + P_{33} = 1 \Rightarrow P_{33} = 0{,}6$$

$$\Rightarrow P = \begin{pmatrix} 0{,}1 & 0{,}9 & 0 \\ 0\ 1 & 0{,}4 & 0{,}5 \\ 0{,}1 & 0{,}3 & 0{,}6 \end{pmatrix}$$

<u>Graphe</u>



$$P(X_1 = 2) = P(X_1 = 2, X_0 = 1) + P(X_1 = 2 \mid X_0 = 2) + P(X_1 = 2 \mid X_0 = 3)$$

$= P(X_0 = 1) \cdot P(X_1 = 2 | X_0 = 1) + P(X_0 = 2) \; P(X_1 = 2 | X_0 = 2)$

$+ P(X_0 = 3) \; P(X_1 = 2 | X_0 = 3)$

$= 0{,}2 \times 0{,}9 + 0{,}4 \times 0{,}5 + 0{,}4 \times 0{,}3$

$= 0{,}18 + 0{,}2 + 0{,}12 = 0{,}18 + 0{,}32 = 0{,}5$

$\Rightarrow \boxed{P(X_1 = 2) = 0{,}5}$

3) $A \sim B$ et $B \sim C$ classe transitoire

4) $\pi = (p_1, p_2, p_3)$ stationnaire $\Leftrightarrow \pi P = \pi$

$$\Leftrightarrow \begin{cases} p_1 \times 0{,}1 + p_2 \times 0{,}1 + p_3 \times 0{,}1 = p_1 \\ p_1 \times 0{,}9 + p_2 \times 0{,}4 + p_3 \times 0{,}3 = p_2 \\ p_2 \times 0{,}5 + p_3 \times 0{,}6 = p_3 \end{cases}$$

$$\iff p_1 \times 0{,}1 + \underbrace{(p_2 + p_3)}_{=(1-p_1)} \times 0{,}1 = p_1 \Rightarrow \underline{p_1 = 0{,}1}$$

$$0{,}6 \times p_2 = 0{,}3 \times p_3 + 0{,}09$$

$$0{,}5 \times p_2 = 0{,}4 \times p_3 \Rightarrow p_3 = \frac{5}{4} p_2$$

$$\Rightarrow 0{,}6 \times p_2 = \frac{0{,}3 \times 5}{4} p_2 + 0{,}09$$

$$\Rightarrow 0{,}3 \times p_2 \left(2 - \frac{5}{4}\right) = 0{,}3 \times \frac{3}{4} p_2 = 0{,}09$$

$$\Rightarrow p_2 = \frac{0{,}36}{0{,}9} = 0{,}4 \Rightarrow \underline{p_2 = 0{,}4}$$

$$\Rightarrow \underline{p_3 = 0{,}5}$$

$\Rightarrow$ <u>Répartition des clients à long terme</u> ! A : 10%, B : 40%, C : 50%

III

i) $X \sim Exp(\theta, \nu)$

$$C_X =]\nu, +\infty[$$

$$f_X(x) = \begin{cases} \theta\, e^{-\theta(x-\nu)} & \forall x \in C_X \\ 0 & \text{ailleurs} \end{cases}$$

$$F_X(x) = \theta \int_\nu^x e^{-\theta(t-\nu)} dt. = \theta\, e^{\theta\nu} \left[-\frac{e^{-\theta t}}{\theta} \right]_\nu^x = 1 - e^{-\theta(x-\nu)}$$

$\forall x \in C_X$

$$F_X(x) = \begin{cases} 1 - e^{-\theta(x-\nu)} & \forall x \in C_X \\ 0 & \text{ailleurs} \end{cases}$$

Fonction de Fiabilité : $\phi_X(x_0) = 1 - F_X(x_0) = e^{-\theta(x_0-\nu)}$ $\quad (x_0 > \nu)$

iii) T = temps d'attente avant la 1° panne

$T \sim \text{Exp}(\theta, 0)$

1. $f_T(t) = \theta e^{-\theta t} \qquad t > 0$

$P(T > 5) = \phi_T(5) = e^{-5\theta} = 0{,}018 \Rightarrow -5\theta = \ln(0{,}018) = -4$

$\Rightarrow \underline{\theta} = \frac{4}{5} = \underline{0{,}8}$

$\underline{f_T(t) = 0{,}8\, e^{-0{,}8 \times t} \qquad (t > 0)}$

2. $E(T) = \frac{1}{\theta} = \frac{10}{8} = \frac{5}{4} = 1 + \frac{1}{4} = \underline{1{,}25}$ ans

$\underline{F_T(t) = 1 - e^{-0{,}8 \times t} \qquad (t > 0)}$

3. $P(T>6 \mid T>3) = \dfrac{P(T>6)}{P(T>3)} = \dfrac{e^{-0,8\times 6}}{e^{-0,8\times 3}} = e^{-0,8\times 3}$

$= e^{-0,24} \approx \underline{78\%}$

IV

X = nombre de pièces défectueuses parmi 100

- X suit $\mathcal{B}(100;\ 0,01)$
  ↳ probabilité pour une pièce d'être défectueuse

$P(X>3) = 1 - P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$\approxeq \underline{0,99}$

- X suit $\mathcal{P}(1)$

$P(X=k) = e^{-1}\dfrac{1^k}{k!} = \dfrac{e^{-1}}{k!}$

$\Rightarrow \quad P(X > 3) = e^{-1} + e^{-1} + \frac{e^{-1}}{2} = e^{-1}(2,5) = 0,368 \times 2,5$

$= \underline{0,92}$

2. Probabilité d'apparition de 3 lors d'un tirage : 0,1.

- Y suit $\mathcal{B}(10\,000 ; 0,1)$

$$P(Y < 950) = \sum_{k=0}^{949} C_{10000}^{k} (0,1)^{k} (0,9)^{10\,000-k}$$

- Y peut être approchée par $\mathcal{N}(1000 ; 900)$

$$P(Y \leq 950) = P\left(Z \leq \frac{950-1000}{30}\right) \quad \text{où } Z = \frac{Y-1000}{30} \text{ suit } \mathcal{N}(0,1)$$

$$= P\left(Z \leq -\frac{5}{3}\right) = 1 - P\left(Z \leq \frac{5}{3}\right) = 1 - P(Z \leq 1,66)$$

$$= 1 - 0,95154 \simeq 0,05 = 5\%$$