Analyse numérique, TD n°7

Exercice 1

Soit le système suivant :

$$\begin{matrix}a\_{11}x\_{1}+a\_{12}x\_{2}=&b\_{1}\\a\_{21}x\_{1}+a\_{22}x\_{2}=&b\_{2}\end{matrix}$$

Cela revient à résoudre Ax=b avec :

$$A= \begin{matrix}a\_{11}&a\_{12}\\a\_{21}&a\_{22}\end{matrix} et b= \begin{matrix}b\_{1}\\b\_{2}\end{matrix}$$

1. Matrices d’itération
2. Jacobi

$$M= \left[\begin{matrix}a\_{11}&0\\0&a\_{22}\end{matrix}\right] et N= \left[\begin{matrix}0&-a\_{12}\\-a\_{21}&0\end{matrix}\right] $$

$$=>J=-\left[\begin{matrix}\frac{1}{a\_{11}}&0\\0&\frac{1}{a\_{22}}\end{matrix} \right].\left[\begin{matrix}0&a\_{12}\\a\_{21}&0\end{matrix}\right]= -\left[\begin{matrix}0&\frac{a\_{12}}{a\_{11}}\\\frac{a\_{21}}{a\_{22}}&0\end{matrix}\right]$$

1. Gauss Seidel

$$M= \left[\begin{matrix}a\_{11}&0\\a\_{21}&a\_{22}\end{matrix}\right] et N= \left[\begin{matrix}0&-a\_{12}\\0&0\end{matrix}\right]$$

$$=>G=-\frac{1}{a\_{11}a\_{22}}\left[\begin{matrix}a\_{22}&0\\-a\_{21}&a\_{11} \end{matrix} \right].\left[\begin{matrix}0&a\_{12}\\0&0\end{matrix}\right]=\frac{1}{a\_{11}a\_{22}} \left[\begin{matrix}0&-a\_{22}a\_{12}\\0&a\_{21}a\_{12}\end{matrix}\right]$$

1. Rayons spectraux
2. Jacobi

Les valeurs propres de J sont$ \pm \sqrt{\frac{a\_{12}a\_{21}}{a\_{11}a\_{22}}}$.

* $ρ\left(J\right)=\sqrt{\frac{a\_{12}a\_{21}}{a\_{11}a\_{22}}}.$
1. Gauss Seidel

Les valeurs propres sont$ 0 et\frac{a\_{12}a\_{21}}{a\_{11}a\_{22}}$.

* $ρ\left(G\right)=\left|\frac{a\_{12}a\_{21}}{a\_{11}a\_{22}}\right|.$

Exercice 3.3

Soit $M=\left(\begin{matrix}A&B\\B&A\end{matrix}\right)$

1. Matrices d’itération
2. Jacobi

$$M= \left[\begin{matrix}A&\left(0\right)\\\left(0\right)&A\end{matrix}\right] et N= -\left[\begin{matrix}\left(0\right)&B\\B&\left(0\right)\end{matrix}\right] $$

$$=>J=-\left[\begin{matrix}A^{-1}&0\\0&A^{-1}\end{matrix} \right].\left[\begin{matrix}\left(0\right)&B\\B&\left(0\right)\end{matrix}\right]= -\left[\begin{matrix}0&A^{-1}B\\A^{-1}B&0\end{matrix}\right]=-\left[\begin{matrix}0&B\\B&0\end{matrix}\right]$$

1. Gauss Seidel

$$M= \left[\begin{matrix}A&\left(0\right)\\B&A\end{matrix}\right] et N= -\left[\begin{matrix}\left(0\right)&B\\\left(0\right)&\left(0\right)\end{matrix}\right]$$

$$=>G=-\left(A^{-1}\right)^{2}\left[\begin{matrix}A&0\\B&A \end{matrix} \right].\left[\begin{matrix}\left(0\right)&B\\\left(0\right)&\left(0\right)\end{matrix}\right]=\left[\begin{matrix}\left(0\right)&-B\\\left(0\right)&B^{2}\end{matrix}\right]$$

1. Avec relaxation

$$x^{\left(k+1\right)}=\left(1-ω\right).\left(I+ωD^{-1}L\right)^{-1}x^{\left(k\right)}-ω\left(I+ωD^{-1}L\right)^{-1}D^{-1}Ux^{\left(k\right)}+ω\left(I+ωD^{-1}L\right)^{-1}D^{-1}b$$

$$G\_{ω}=\left(I+ωD^{-1}L\right)^{-1}\left[\left\{1-ω\right\}I-ωD^{-1}U\right]$$

$$or D=I\_{4}=>\left(I+ωD^{-1}L\right)^{-1}=\left(\begin{matrix}1&0\\-ωk&1\end{matrix}\right)$$

$$G\_{ω}=\left[\begin{matrix}1&0\\-ωB&1\end{matrix}\right]\left(\left[\begin{matrix}1-ω&0\\0&1-ω\end{matrix}\right]-\left[\begin{matrix}0&ωB\\0&0\end{matrix}\right]\right)$$

$$G\_{ω}=\left[\begin{matrix}1&0\\-ωB&1\end{matrix}\right]\left[\begin{matrix}1-ω&-ωB\\0&1-ω\end{matrix}\right]=\left[\begin{matrix}1-ω&-ωB\\ωB\left(w-1\right)&1-ω+ω²B²\end{matrix}\right] $$

1. Application

$$A=\left[\begin{matrix}1&0&-\frac{1}{4}&-\frac{1}{4}\\0&1&-\frac{1}{4}&-\frac{1}{4}\\-\frac{1}{4}&-\frac{1}{4}&1&0\\-\frac{1}{4}&-\frac{1}{4}&0&1\end{matrix}\right] ; b= -\frac{1}{2}\left[\begin{array}{c}1\\1\\1\\1\end{array}\right] ;$$

1. Jacobi

$$x^{\left(k+1\right)}=Jx^{\left(k\right)}+ D^{-1}b $$

$$=-\left[\begin{matrix}0&B\\B&0\end{matrix}\right]\left[\begin{array}{c}x\_{1}\\x\_{2}\\x\_{3}\\x\_{4}\end{array}\right]+\frac{1}{2}\left[\begin{array}{c}1\\1\\1\\1\end{array}\right]$$

$$=\left[\begin{array}{c}-\frac{x\_{3}+x\_{4}}{4}\\-\frac{x\_{3+x\_{4}}}{4}\\-\frac{x\_{1}+x\_{2}}{4}\\-\frac{x\_{1}+x\_{2}}{4}\end{array}\right] \frac{1}{2}\left[\begin{array}{c}1\\1\\1\\1\end{array}\right]$$

$$x^{0}= \left[\begin{array}{c}0\\0\\0\\0\end{array}\right] ;x^{1}= \frac{1}{2}\left[\begin{array}{c}1\\1\\1\\1\end{array}\right] ;x^{2}= \frac{1}{4}\left[\begin{array}{c}1\\1\\1\\1\end{array}\right]$$

1. Gauss Seidel

$$x^{\left(k+1\right)}=Gx^{\left(k\right)}+ \left(D+L\right)^{-1}b $$

$$=\left[\begin{matrix}0&-B\\0&B²\end{matrix}\right]\left[\begin{array}{c}x\_{1}\\x\_{2}\\x\_{3}\\x\_{4}\end{array}\right]+\left[\begin{matrix}I\_{2}&0\\-B&I\_{2}\end{matrix}\right].\frac{1}{2}\left[\begin{array}{c}1\\1\\1\\1\end{array}\right]$$

$$=\left[\begin{array}{c}-\frac{x\_{3}+x\_{4}}{4}\\-\frac{x\_{3+x\_{4}}}{4}\\\frac{x\_{3}+x\_{4}}{16}\\\frac{x\_{3}+x\_{4}}{16}\end{array}\right]+\left[\begin{array}{c}\frac{1}{2}\\\frac{1}{2}\\\frac{3}{4}\\\frac{3}{4}\end{array}\right]$$

$$x^{0}= \left[\begin{array}{c}0\\0\\0\\0\end{array}\right] ;x^{1}=\frac{1}{4} \left[\begin{array}{c}2\\2\\3\\3\end{array}\right] ;x^{2}=$$