DS Algèbre Ing 1 2010

Titre de la note 30/11/2012

I i) $P_A(t) = (t-\sqrt{5})^6 (t-1)^4$

$m_A(t) = (t-\sqrt{5})^4 (t-1)^4$

A d'ordre 10×10

valeurs propres $\lambda_1 = \sqrt{5}$ $\lambda_2 = 1$

$$J_1 = \begin{bmatrix} \sqrt{5} & 1 & 0 & 0 \\ 0 & \sqrt{5} & 1 & 0 \\ 0 & 0 & \sqrt{5} & 1 \\ 0 & 0 & 0 & \sqrt{5} \end{bmatrix} \qquad J_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

d'ordre 4×4 d'ordre 4×4

Reste la place pour une matrice d'ordre 2

ou 2 matrices d'ordre 1

$$\begin{pmatrix} J_1 & & \\ & \begin{pmatrix} \sqrt{5} & 1 \\ 0 & \sqrt{5} \end{pmatrix} & \\ & & J_2 \end{pmatrix} \quad \text{ou} \quad \begin{pmatrix} J_1 & & \\ & \begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} & \\ & & J_2 \end{pmatrix}$$

$$\text{ou} \quad \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \quad \text{ou} \quad \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

$$\text{ou} \quad \begin{pmatrix} J_1 & & & \\ & (\sqrt{5}) & & 0 \\ & & J_2 & \\ & 0 & & (1) \end{pmatrix}$$

ii) $\dim E = 14 \qquad f : E \to E$

$$m_f(t) = P_1(t)\, P_2(t)^2\, P_3(t)^3$$

avec $\begin{cases} P_1(t) = t^2 + 1 \\ P_2(t) = t^2 + \sqrt{2} \\ P_3(t) = t^2 + \sqrt{3} \end{cases}$

$$P_1(t)^{m_{11}} = P_1(t) = t^2 + 1$$

$$P_2(t)^{m_{21}} = \left(P_2(t)\right)^2 = t^4 + 2\sqrt{2}\, t^2 + 2$$

$$P_2(t)^{m_{22}} = t^2 + \sqrt{2}$$

$$P_3(t)^{m_{31}} = \left(t^2 + \sqrt{3}\right)^3 = t^6 + 3\sqrt{3}\, t^4 + 9 t^2 + 3\sqrt{3}$$

$$P_3(t)^{m_{32}} = \left(t^2 + \sqrt{3}\right)^2 = t^4 + 2\sqrt{3}\, t^2 + 3$$

$$P_3(t)^{m_{33}} = t^2 + \sqrt{3}$$

$$C_{11} = C(t^2+1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$C_{21} = C(t^4 + 2\sqrt{2}\,t^2 + 2) = \begin{pmatrix} 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2\sqrt{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$C_{22} = C(t^2 + \sqrt{2}) = \begin{pmatrix} 0 & -\sqrt{2} \\ 1 & 0 \end{pmatrix}$$

$$C_{31} = C(t^6 + 3\sqrt{3}\,t^4 + 9\,t^2 + 3\sqrt{3}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -3\sqrt{3} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -9 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3\sqrt{3} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$C_{32} = C\left(t^4 + 2\sqrt{3}\,t^2 + 3\right) = \begin{pmatrix} 0 & 0 & 0 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2\sqrt{3} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$C_{33} = C\left(t^2 + \sqrt{3}\right) = \begin{pmatrix} 0 & -\sqrt{3} \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} C_{11} & & & \\ & C_{21} & & \\ & & C_{22} & \\ & & & C_{31} \end{pmatrix} \quad \text{ou} \quad \begin{pmatrix} C_{11} & & & \\ & C_{21} & & \\ & & C_{31} & \\ & & & C_{33} \end{pmatrix}$$

II i) f est un produit scalaire sur $\mathbb{C}^4$

car $f(\lambda X_1 + \mu X_2, Y) = \lambda f(X_1, Y) + \mu f(X_2, Y)$

$f(X,Y) = \overline{f(Y,X)}$

$f(X,X) = \sum_{i=1}^{4} |x_i|^2 \geq 0$ et $f(X,X) = 0 \iff X = 0$

norme associée $\sqrt{f(X,X)} = \|X\| = \sqrt{\sum_{i=1}^{4} |x_i|^2}$

$T(x_1, x_2, x_3, x_4) = T(X) = (x_1 - i x_3, x_2, x_3 + i x_1, x_4)$

2) Adjoint de T

$f(TX, Y) = f(X, T^* Y)$

$f(TX, Y) = (x_1 - i x_3)\overline{y_1} + x_2 \overline{y_2} + (x_3 + i x_1)\overline{y_3} + x_4 \overline{y_4}$

$$= x_1(\overline{y_1} + i\overline{y_3}) + x_2\overline{y_2} + x_3(-i\overline{y_1} + \overline{y_3}) + x_4\overline{y_4}$$

$$= f(X, T^*Y)$$

$$\Rightarrow \overline{T^*Y} = (\overline{y_1} + i\overline{y_3}, \overline{y_2}, -i\overline{y_1} + \overline{y_3}, \overline{y_4})$$

$$T^*Y = (y_1 - iy_3, y_2, iy_1 + y_3, y_4)$$

$$\underline{T^*X = (x_1 - ix_3, x_2, ix_1 + x_3, x_4)} = TX \Rightarrow T = T^*$$

b) $$A = \begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & 0 \\ i & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = B.$$

$T = T^*$ est hermitienne

c) forme quadratique hermitienne associée à A :

$$f(TX, X) = (x_1 - i x_3)\overline{x_1} + x_2 \overline{x_2} + (x_3 + i x_1)\overline{x_3} + x_4 \overline{x_4}$$

$$= |x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2 - i x_3 \overline{x_1} + i x_1 \overline{x_3}$$

$$A = \begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & 0 \\ i & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$L_3' \leftarrow -i L_1 + L_3$$

$$A'' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -i & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$C_3' = i\,C_1 + C_3$$

$$A'' = A' \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= H$$

$\Rightarrow$ $H = C^* A C$ avec

$$C = \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

iii) a) $$V = \begin{pmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

V réelle $\Rightarrow$ $$V^* = {}^tV = \begin{pmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$
{}^tV V = I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow V \text{ unitaire}
$$

$V \neq V^* = {}^tV \Rightarrow V$ n'est pas hermitienne

V unitaire $\Rightarrow V$ normale

V unitaire $\Leftrightarrow V$ orthogonale

b) $\|u\|_2^2 = \dfrac{(1+i)(1-i)}{2} + \dfrac{(1-i)(1+i)}{2} + 1 = 2\,\dfrac{1-i^2}{2} + 1 = 3$

$\Rightarrow \|u\|_2 = \sqrt{3}$

$$V(u) = \begin{pmatrix} \frac{1+i}{\sqrt{2}} \cos\phi + \frac{1-i}{\sqrt{2}} \sin\phi \\ 0 \\ -\frac{1+i}{\sqrt{2}} \sin\phi + \frac{(1-i)}{\sqrt{2}} \cos\phi \\ 1 \end{pmatrix}$$

$$\| V(u) \|_2^2 = \left(\frac{1+i}{\sqrt{2}} \cos\phi + \frac{1-i}{\sqrt{2}} \sin\phi \right)\left(\frac{1-i}{\sqrt{2}} \cos\phi + \frac{1+i}{\sqrt{2}} \sin\phi \right)$$

$$+ \left(-\frac{1+i}{\sqrt{2}} \sin\phi + \frac{1-i}{\sqrt{2}} \cos\phi \right)\left(-\frac{1-i}{\sqrt{2}} \sin\phi + \frac{1+i}{\sqrt{2}} \cos\phi \right)$$

$$+ 1 = \cos^2\phi + \frac{(1-i)^2}{2} \sin\phi\cos\phi + \sin^2\phi + \frac{(1+i)^2}{2} \sin\phi\cos\phi$$

$$+ \sin^2\phi + \cos^2\phi - \frac{(1-i)^2}{2} \sin\phi\cos\phi - \frac{(1+i)^2}{2} \sin\phi\cos\phi$$

$$+ 1$$

$= 3 \quad \Rightarrow \quad \|V(u)\|_2 = \sqrt{3}$

$\|V(u)\|_2 = \|u\|_2$

<u>Théorème</u>

$\|V(u)\|_2^2 = \beta(VU, VU) = \beta(U, V^*VU) = \beta(U,U) = \|u\|_2^2$

puisque V est unitaire

c) $\|V(u)\|_2 = \|u\|_2 \quad \Rightarrow \quad \|V\|_2 = 1$

<u>III</u> $\quad \tilde{A}^* = \tilde{A} \quad \Rightarrow \quad \tilde{A}$ est hermitienne

$$\det(\tilde{A} - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & i\sqrt{5} \\ 0 & -i\sqrt{5} & 5-\lambda \end{vmatrix}$$

$$= (3-\lambda)\left[(1-\lambda)(5-\lambda) - 5\right]$$

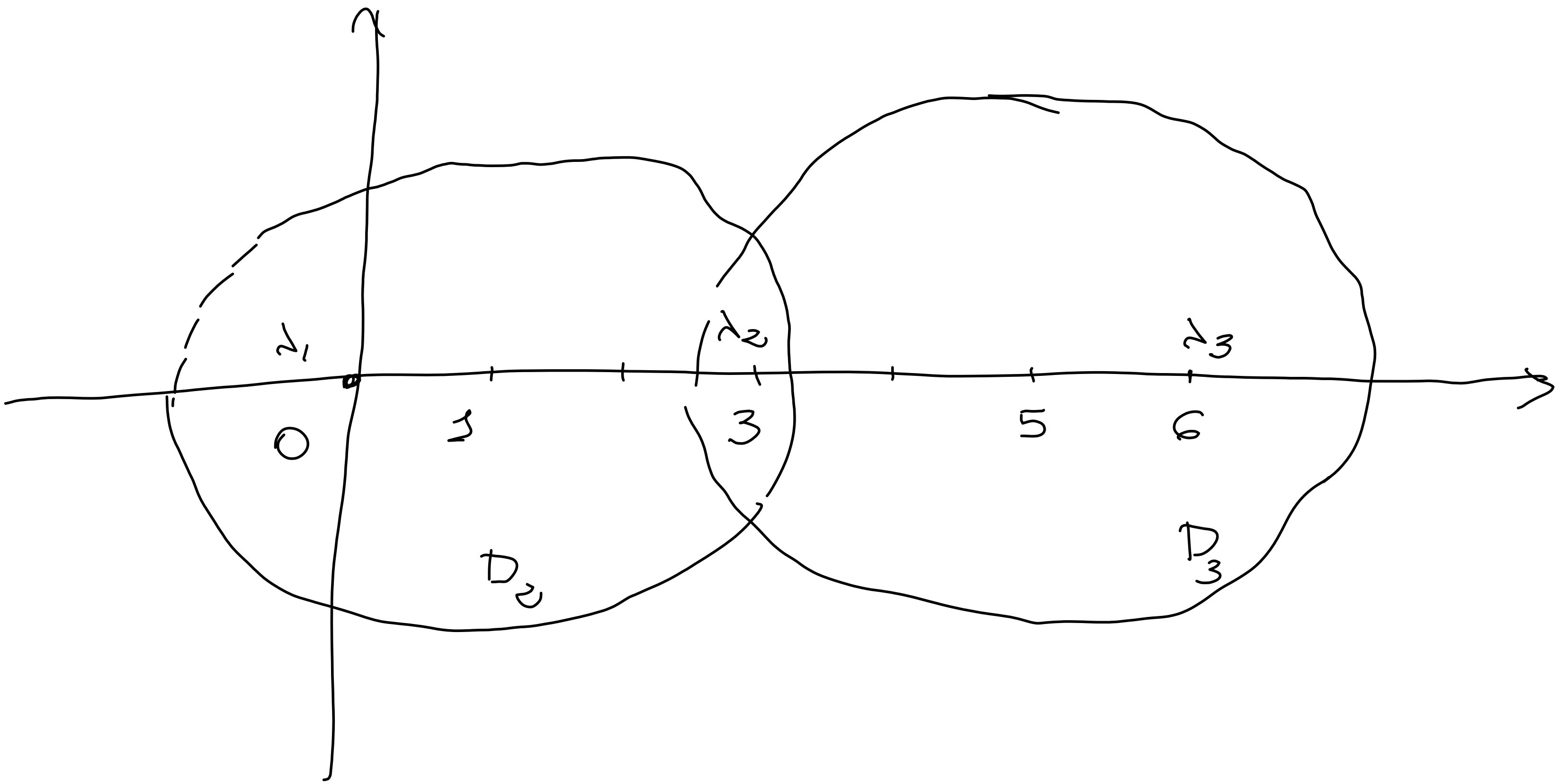
$$= (3-\lambda)(\lambda^2 - 6\lambda) = \lambda(3-\lambda)(\lambda-6)$$

valeurs propres : $\lambda_1 = 0$, $\lambda_2 = 3$, $\lambda_3 = 6$

$D_1 = \{ z \in \mathbb{C} : |z-3| \leq 0 \} = \{3\}$

$D_2 = \{ z \in \mathbb{C} : |z-1| \leq \sqrt{5} \}$

$D_3 = \{ z \in \mathbb{C} : |z-5| \leq \sqrt{5} \}$



$Sp(\tilde{A}) \subset D_2 \cup D_3$ et $3 \in D_2 \cup D_3$

(ii)

a) $B = \begin{pmatrix} 1 & i & -1 \\ 1 & -1 & i \end{pmatrix}$ $\quad B^* = \begin{pmatrix} 1 & 1 \\ -i & -1 \\ -1 & -i \end{pmatrix}$

$C = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ symétrique $\Rightarrow$ hermitienne

b) $\det(C - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1$

$= \lambda^2 - 6\lambda + 8$

$\Delta' = 9 - 8 = 1$ $\Rightarrow$ $\lambda_1 = 3 + 1 = 4$

$\lambda_2 = 3 - 1 = 2$

$\Rightarrow \det(C - \lambda I) = (\lambda - 4)(\lambda - 2)$

$\Rightarrow$ valeurs propres $\lambda_1 = 2$ $\lambda_2 = 4$

$$\rho(C) = \max|\lambda_i| = 4$$

$$\|B\|_2 = \sqrt{\rho(BB^*)} = \sqrt{\rho(C)} = 2$$

$$\Rightarrow \|B\|_2 = 2$$