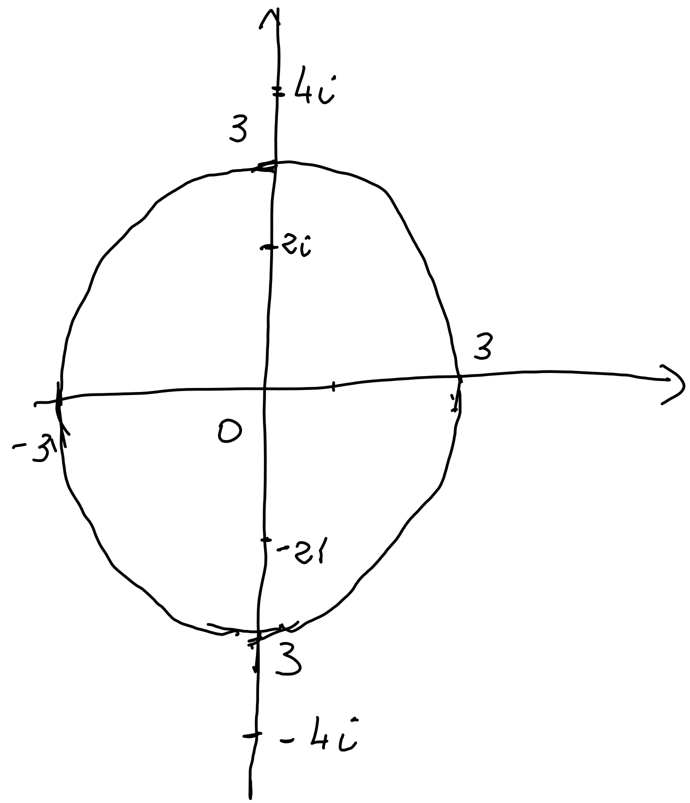
DS Analyse 1 - Janvier 2013

Titre de la note 09/02/2013

$\boxed{\text{I} \quad i)}$ $$I = \oint_{\mathcal{C}} \frac{z+1}{(z^2+16)(z^2+4)}$$

$\mathcal{C}$ = cercle de centre 0
rayon 3



pôles : $4i$, $-4i$, $2i$, $-2i$

1) Application du Théorème des Résidus

$$f(z) = \frac{z+1}{(z^2+16)(z^2+4)}$$

$$I = 2i\pi \sum_{\substack{\text{pôles intérieurs} \\ \text{à } \mathcal{C}}} \text{Res}\, f = \text{Res}(f, 2i) + \text{Res}(f, -2i)$$

$2i$ et $-2i$ sont des pôles simples.

$$\text{Res}(f, 2i) = \lim_{z \to 2i} (z-2i) f(z) = \lim_{z \to 2i} \frac{z+1}{(z^2+16)(z+2i)} = \frac{1+2i}{(16-4)\, 4i}$$

$$\text{Res}(f, 2i) = \frac{1}{12}\left(-\frac{1}{4} i + \frac{1}{2}\right) = \frac{1}{48}(2-i)$$

$$\text{Res}(f, -2i) = \lim_{z \to -2i} (z+2i) f(z) = \lim_{z \to -2i} \frac{z+1}{(z^2+16)(z-2i)} = \frac{1-2i}{(16-4)(-4i)}$$

$$\text{Res}(f, -2i) = \frac{1}{12}\left(\frac{1}{2} + \frac{1}{4} i\right) = \frac{1}{48}(2+i)$$

$$\Rightarrow I = \frac{2i\pi}{48}\left(2-i+2+i\right) = \frac{8}{48} i\pi = \frac{1}{6} i\pi \Rightarrow \boxed{I = \frac{i\pi}{6}}$$

2) formule intégrale de Cauchy

$$\frac{1}{z^2+4} = \frac{b_1}{z+2i} + \frac{b_2}{z-2i}$$

on × par $(z-2i)$ puis $z = 2i$ $\qquad \frac{1}{4i} = b_2 = -\frac{i}{4}$

on × par $(z+2i)$ puis $z = -2i$ $\qquad -\frac{1}{4i} = b_1 = \frac{i}{4}$

$$\Rightarrow \frac{1}{z^2+4} = \frac{i}{4(z+2i)} - \frac{i}{4(z-2i)} = \frac{i}{4}\left(\frac{1}{z+2i} - \frac{1}{z-2i}\right)$$

$$\Rightarrow I = \frac{i}{4}\left(\oint_{\mathcal{C}} \frac{g(z)}{z+2i}\,dz - \oint_{\mathcal{C}} \frac{g(z)}{z-2i}\,dz\right) \quad \text{avec } g(z) = \frac{z+1}{z^2+16}$$

Formule intégrale de Cauchy : $g(z_0) = \frac{1}{2i\pi} \oint_{\mathcal{C}} \frac{g(z)}{z-z_0}\,dz$

où z_0 est un point intérieur à $\mathcal{C}$

$$\Rightarrow I = \frac{i}{4} \cdot 2i\pi \left(g(-2i) - g(2i) \right) = -\frac{\pi}{2}\left(\frac{1-2i}{12} - \frac{1+2i}{12} \right)$$

$$\Rightarrow I = \frac{4i\pi}{2\times 12} = \frac{i\pi}{6} \Rightarrow \boxed{I = \frac{i\pi}{6}}$$

$\boxed{I(ii)}$

$$\int_{-\infty}^{+\infty} f(u)\, g(x-u)\, du = h(x) \qquad (D0)$$

avec $g(x) = \dfrac{x}{x^2+4}$, $h(x) = \dfrac{x}{x^2+9}$

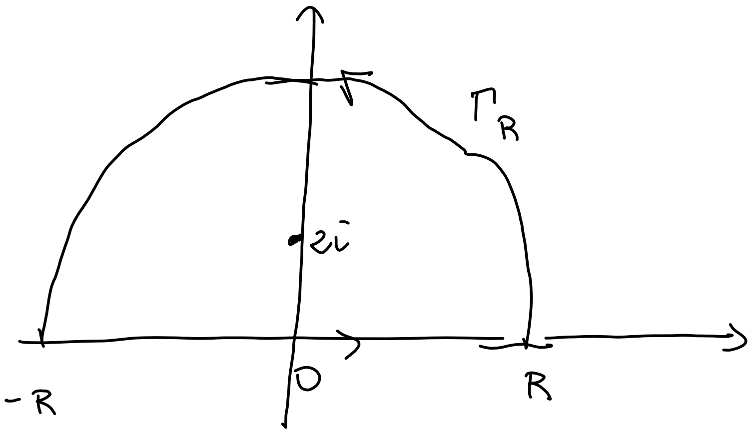
a) $$\mathcal{F}_s(g)(\alpha) = \int_0^{+\infty} g(x) \sin(\alpha x)\, dx = \frac{1}{2} \int_{-\infty}^{+\infty} g(x) \sin(\alpha x)\, dx$$

car la fonction $g(x) \sin(\alpha x)$ est paire

$$\Rightarrow \mathcal{F}_s(g)(\alpha) = \frac{1}{2} \operatorname{Im}\left(\int_{-\infty}^{+\infty} g(x)\, e^{i\alpha x}\, dx \right)$$

$$\mathcal{F}_s(g)(-\alpha) = -\mathcal{F}_s(g)(\alpha) \quad \Rightarrow \quad \text{on suppose } \alpha > 0$$

Contour de Jordan :



$$\mathcal{C}_R = \Gamma_R \cup [-R, R]$$

Théorème des Résidus : pôles de g : $2i, -2i$ pôles simples

$$\oint_{\mathcal{C}_R} g(z)\, e^{i\alpha z}\, dz = 2i\pi \sum_{\substack{\text{pôles intérieurs} \\ \text{à } \mathcal{C}_R}} \operatorname{Res}\left(g(z)\, e^{i\alpha z}\right)$$

Pour R assez grand, seul le pôle $2i$ est à l'intérieur de $\mathcal{C}_R$

$$\Rightarrow \oint_{\mathcal{C}_R} g(z)\, e^{i\alpha z}\, dz = 2i\pi \ \mathrm{Res}\left(g(z)e^{i\alpha z}, 2i\right) = 2i\pi \lim_{z\to 2i} \frac{z\, e^{i\alpha z}}{z+2i}$$

$$= 2i\pi \ \frac{2i\ e^{-2\alpha}}{4i} = i\pi\, e^{-2\alpha}$$

$$\oint_{\mathcal{C}_R} g(z)\, e^{i\alpha z}\, dz = \int_{-R}^{R} g(x)\, e^{i\alpha x}\, dx + \int_{\Gamma_R} g(z)\, e^{i\alpha z}\, dz = i\pi\, e^{-2\alpha}$$

$$\lim_{R\to +\infty} \int_{-R}^{R} g(x)\, e^{i\alpha x}\, dx = \int_{-\infty}^{+\infty} g(x)\, e^{i\alpha x}\, dx$$

Pour l'autre intégrale, on vérifie les conditions du Lemme de Jordan:

Il faut vérifier. $\lim\limits_{|z|\to+\infty} |g(z)| = 0$

$$g(z) = \frac{z}{z^2+4} \Rightarrow g(z) = \frac{1}{z} \frac{1}{1+\frac{4}{z^2}}$$

$$\Rightarrow |g(z)| = \frac{1}{|z|} \frac{1}{\left|1+\frac{4}{z^2}\right|}$$

$$\lim_{|z|\to+\infty} \left|\frac{4}{z^2}\right| = \lim_{|z|\to+\infty} \frac{4}{|z|^2} = 0 \Rightarrow \lim_{|z|\to+\infty} \frac{4}{z^2} = 0 \Rightarrow \lim_{|z|\to+\infty} \left(1+\frac{4}{z^2}\right) = 1$$

$$\Rightarrow \lim_{|z|\to\infty} \left|1+\frac{4}{z^2}\right| = \left|\lim_{|z|\to+\infty} \left(1+\frac{4}{z^2}\right)\right| = |1| = 1$$

$$\lim_{|z|\to+\infty} \frac{1}{|z|} = 0 \Rightarrow \lim_{|z|\to+\infty} |g(z)| = 0$$

Le lemme de Jordan entraîne $\lim_{R \to +\infty} \oint_{\Gamma_R} g(z)\, e^{i\alpha z}\, dz = 0$

On déduit : $\displaystyle\int_{-\infty}^{+\infty} g(x)\, e^{i\alpha x}\, dx = i\pi\, e^{-2\alpha}$

$$\Rightarrow \mathcal{F}_s(g)(\alpha) = \frac{1}{2} \operatorname{Im}\left(\int_{-\infty}^{+\infty} g(x)\, e^{i\alpha x}\, dx \right) = \frac{\pi}{2} e^{-2\alpha}$$

a') Par analogie

$$\mathcal{F}_s(h)(\alpha) = \frac{1}{2} \operatorname{Im}\left(\int_{-\infty}^{+\infty} h(x)\, e^{i\alpha x}\, dx \right) = \frac{1}{2} \operatorname{Im}\left(\oint_{\mathcal{C}_R} h(z)\, e^{i\alpha z}\, dz \right)$$

$$\Rightarrow \mathcal{F}_s(h)(\alpha) = \frac{1}{2} \operatorname{Im}\left(2i\pi \operatorname{Res}\left(h(z)\, e^{i\alpha z}, 3i\right)\right) = \operatorname{Im}\left(i\pi \frac{3i\, e^{-3\alpha}}{6i} \right)$$

$$\Rightarrow \quad \mathcal{F}_s(h)(\alpha) = \operatorname{Im}\left(\frac{i\pi}{2} e^{-3\alpha}\right) = \frac{\pi}{2} e^{-3\alpha}$$

b) $$\mathcal{F}_s(f * g)(\alpha) = \mathcal{F}_s(h)(\alpha) = 2\,\mathcal{F}_s(f)^{(\alpha)}\,\mathcal{F}_s(g)(\alpha)$$

$$\Rightarrow \quad \frac{\pi}{2} e^{-3\alpha} = 2\,\mathcal{F}_s(f)(\alpha) \cdot \frac{\pi}{2} e^{-2\alpha}$$

$$\Rightarrow \quad \boxed{\mathcal{F}_s(f)(\alpha) = \frac{1}{2} e^{-\alpha}}$$

c) $$f(x) = \frac{2}{\pi} \int_0^{+\infty} \frac{1}{2} e^{-\alpha} \sin(\alpha x)\, d\alpha = \frac{1}{\pi} \int_0^{+\infty} e^{-\alpha} \frac{\left(e^{i\alpha x} - e^{-i\alpha x}\right)}{2i}\, d\alpha$$

$$f(x) = \frac{1}{2i\pi} \int_0^{+\infty} \left(e^{-\alpha(1-ix)} - e^{-\alpha(1+ix)} \right) d\alpha$$

$$f(x) = \frac{1}{2i\pi} \left(\lim_{A \to +\infty} \left[\frac{e^{-\alpha(1-ix)}}{-(1-ix)} \right]_0^A - \lim_{A \to +\infty} \left[\frac{e^{-\alpha(1+ix)}}{-(1+ix)} \right]_0^A \right)$$

$$f(x) = \frac{1}{2i\pi} \left(\frac{1}{1-ix} - \lim_{A \to +\infty} \frac{e^{-A(1-ix)}}{1-ix} - \frac{1}{1+ix} + \lim_{A \to +\infty} \frac{e^{-A(1+ix)}}{1+ix} \right)$$

$$\left| e^{-A(1-ix)} \right| = \left| e^{-A(1+ix)} \right| = e^{-A} \xrightarrow[A \to +\infty]{} 0$$

d'où $$f(x) = \frac{1}{2i\pi} \left(\frac{1}{1-ix} - \frac{1}{1+ix} \right) = \frac{1}{2i\pi} \left(\frac{1+ix-1+ix}{1+x^2} \right)$$

$$\boxed{f(x) = \frac{x}{1+x^2}}$$

II $\dfrac{d^2 f(t)}{dt^2} + 6 \dfrac{df(t)}{dt} + 8 f(t) = 3 \exp(-t)$ (P)

$f'(0) = 0 \qquad f(0) = 1$

a) $$\mathcal{L}(e^{-t})(p) = \int_0^{+\infty} e^{-t} e^{-pt}\,dt = \int_0^{+\infty} e^{-t(1+p)}\,dt = \lim_{X \to +\infty} \left[-\frac{e^{-t(1+p)}}{1+p} \right]_0^X$$

$$= -\lim_{X \to +\infty} \frac{e^{-X(1+p)}}{1+p} + \frac{1}{1+p}$$

$\lim\limits_{X \to +\infty} \dfrac{e^{-X(1+p)}}{1+p}$ existe et $= 0 \iff \mathrm{Re}(1+p) > 0 \iff \mathrm{Re}\,p > -1$

$\Rightarrow \mathcal{L}(e^{-t})(p)$ existe uniquement pour $\mathrm{Re}\,p > -1$

et $\boxed{\mathcal{L}(e^{-t})(p) = \dfrac{1}{p+1} \quad \text{pour} \quad \mathrm{Re}(p) > -1}$

b) $\mathcal{L}(f')(p) = p\,\mathcal{L}(f)(p) - f(0) = p\,\mathcal{L}(f)(p) - 1$

$\mathcal{L}(f'')(p) = p\,\mathcal{L}(f')(p) - f'(0) = p\,\mathcal{L}(f')(p) = p^2\mathcal{L}(f)(p) - p$

$(P) \Rightarrow \quad p^2\mathcal{L}(f)(p) - p + 6p\,\mathcal{L}(f)(p) - 6 + 8\,\mathcal{L}(f)(p) = \dfrac{3}{1+p}$

$\Rightarrow \quad \mathcal{L}(f)(p)\,(p^2+6p+8) = \dfrac{3}{1+p} + 6 + p = \dfrac{3+6+6p+p+p^2}{1+p} = \dfrac{9+7p+p^2}{1+p}$

$p^2+6p+8 = (p+3)^2+8-9 = (p+3)^2-1 = (p+4)(p+2)$

$\Rightarrow \quad \mathcal{L}(f)(p) = \dfrac{p^2+7p+9}{(p+1)(p+2)(p+4)}$

p^2+7p+9

$\Delta = 49-36 = 13$

$\Rightarrow p = \dfrac{-7 \pm \sqrt{13}}{2}$

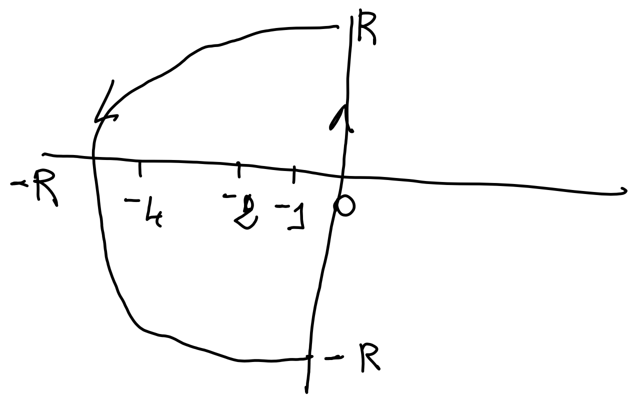
c) pôles simples : $-1, -2, -4$

$$F(p) = \frac{e^{pt}\,(p^2+7p+9)}{(p+1)(p+2)(p+4)}$$

Application du Théorème de Bromwich :

$$f(t) = \mathcal{L}^{-1}\left(\frac{p^2+7p+9}{(p+1)(p+2)(p+4)}\right) = \mathrm{Res}(F,-1) + \mathrm{Res}(F,-2) + \mathrm{Res}(F,-3)$$

Contour de Bromwich



$$\mathrm{Res}(F,-1) = \lim_{p\to -1} (1+p)F(p) = \lim_{p\to -1} \frac{e^{pt}\,(p^2+7p+9)}{(p+2)(p+4)} = \frac{(9-7+1)\,e^{-t}}{3} = e^{-t}$$

$$\operatorname{Res}(F,-2) = \lim_{p\to -2} (p+2)F(p) = \lim_{p\to -2} \frac{e^{pt}(p^2+7p+9)}{(p+1)(p+4)} = \frac{(9-14+4)e^{-2t}}{-2}$$

$$= \frac{1}{2}e^{-2t}$$

$$\operatorname{Res}(F,-4) = \lim_{p\to -4} (p+4)F(p) = \lim_{p\to -4} \frac{e^{pt}(p^2+7p+9)}{(p+1)(p+2)} = \frac{e^{-4t}(9-28+16)}{(-3)(-2)}$$

$$= -\frac{1}{2}e^{-4t}$$

$$\Rightarrow \boxed{f(t) = e^{-t} + \frac{1}{2}\left(e^{-2t} - e^{-4t}\right)}$$